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ALGEBRAIC STRUCTURE OF THE TRUTH-VALUES FOR Ł$_\omega$

This paper is an abstract of the report which was presented on the Polish-Soviet meeting on logic (Moscow, December 6-8, 1986). It is shown that one can consider a lineary-ordered Heyting's and Brouwer's algebras as truth-values for Łukasiewicz's infinite-valued logic's Ł$_\omega$.

1. Algebra H-B-M-S- $<A, 0, 1, \vee, \wedge, \Rightarrow, \dot{-}, \oplus, \sim>$

Let us consider the algebra H-B-M, where $<A, 0, 1, \vee, \wedge>$ is a distributive lattice, $<A, 0, 1, \vee, \wedge, \Rightarrow>$ is a H-algebra (Heyting algebra), $<A, 0, 1, \vee, \wedge, \Rightarrow, \dot{-}>$ is a B-algebra (Brouwer algebra), $<A, 0, 1, \vee, \wedge, \Rightarrow, \dot{-}>$ is H-B-algebra (semi-Boolean algebra) [10], $<A, 0, 1, \vee, \wedge, \sim>$ is a de Morgan algebra, $<A, 0, 1, \vee, \wedge, \Rightarrow, \sim>$ is a symmetrical Heyting algebra [5] and, respectively $<A, 0, 1, \vee, \wedge, \dot{-}, \sim>$ is a symmetrical Brouwer algebra.

Now we define the operation $\dot{\to}$ on the elements of the set A:

$$x \dot{\to} y = (x \Rightarrow y) \oplus \sim (x \dot{-} y),$$

where $\oplus$ is a monoid operations.
Then the operations $\vee$ and $\wedge$ can be expressed as follows:

$$x \vee y = (x \dot{\to} y) \dot{\to} y,$$

$$x \wedge y = \sim (\sim x \vee \sim y).$$

Let A be a linearly-ordered set i.e., a chain with the first element 0 and the last element 1. It is known that the chain with the first and the last elements is a linearly-ordered Heyting algebra [3] or L-algebra [4], i.e. Heyting's algebra satisfying the linearity condition. Let us introduce the following notations: L-algebra in which the first element is the least element is called a linearly-ordered Heyting's algebra (LH-algebra) and L-algebra in which the first element is the greatest one is called a linearly-ordered Brouwer's algebra (LB-algebra).

Since A is a chain then LH- and LB-algebras are defined as follows:

$$LH = \langle A, 0, 1, \vee, \wedge, \Rightarrow \rangle,$$

where

$$x \vee y = max(x, y)$$
$$x \wedge y = min(x, y)$$
$$x \Rightarrow y = \begin{cases} 1, & \text{if} \quad x \leq y \\ y, & \text{if} \quad x > y \end{cases}$$

$$LB = \langle A, 0, 1, \vee, \wedge, \dot{-} \rangle,$$

where

$$x \vee y = min(x, y)$$
$$x \wedge y = max(x, y)$$
$$x \dot{-} y = \begin{cases} 0, & \text{if} \quad x \leq y \\ x, & \text{if} \quad x > y. \end{cases}$$

2. Models for $M_A = \langle A, \dot{\rightarrow}, \sim, \{1\} \rangle$

Let $\underline{M}_A = \langle A, \dot{\rightarrow}, \sim, \{1\} \rangle$ be an abstract logical matrix where the operation $\dot{\rightarrow}$ is defined above and $\{1\}$ is the set of designated elements.

2.1. $\underline{M}^{Ł}_{[0,1]} = \langle [0,1], \rightarrow, \sim, \{1\} \rangle$ be a Łukasiewicz's infinite-valued matrix, where $[0, 1]$ is the set of rational or real numbers such that $0 \leq x \leq 1$; $x \rightarrow y = min(1, 1 - x + y), \sim x = 1 - x$ and $\{1\}$ is the set of designated elements [9]. $\underline{M}^{Ł}_{[0,1]}$ is the correct model (the characteristic matrix) for Łukasiewicz's infinite-valued logics $Ł_\omega$ [2]. It follows from [2] that an algebra for $Ł_\omega$ is MV-algebra, an example of which is

$$MV = \langle [0,1], 0, 1, +, \cdot, \sim \rangle,$$

where $x + y = min(1, x + y), x \cdot y = max(0, x + y - 1), \sim x = 1 - x$.

PROPOSITION 1. *$\underline{M}^{Ł}_{[0,1]} = <[0,1], \rightarrow, \sim, \{1\}>$ and $\underline{M}_{A=[0,1]} = <[0,1], \dot{\rightarrow}, \sim, \{1\}>$ are isomorphic matrices.*

2.2. Let $\underline{M}_\Sigma = <\Sigma, \mapsto, -, \{0^-\}>$ be a logical matrix, where the ordinal type of Σ is $\omega + \omega^*$ i.e.,

$$\Sigma = \{0^+, 1, 2, \ldots, \omega^+, \omega^-, \ldots, -2, -1, 0^-\},$$

$$x \mapsto y = \begin{cases} 0^-, & \text{if } x \leq y \\ y - x, & \text{otherwise} \end{cases}$$

$x = -x, \{0^-\}$ is the set of designated elements [6]. It is important to stress that $\underline{M}_\Sigma$ is a discrete non-standard model for Ł$_\omega$.

PROPOSITION 2. *$\underline{M}_\Sigma = <\Sigma, \mapsto, -, \{0^-\}>$ and $\underline{M}_{A=\Sigma} = <\Sigma, \dot{\rightarrow}, \sim, \{0^-\}>$ are isomorphic matrices.*

3. Factor-semantics for Ł$_\omega$

Factor-semantics for Łukasiewicz n-valued logics Ł$_n$ $(n > 2, n \in N)$ was constructed in [7]. Now, we give the generalization of this method for Ł$_\omega$.

Let $B = \{T, F\}$ be a set of classical truth-values and $\underline{M}^C_2 = <B, \supset, \neg, \{T\}>$ be a two-element Boolean matrix. Let us denote the countably infinite Cartesian power of matrix $\underline{M}^C_2$ by $\underline{M}^C_{\aleph_0} = <B^{\aleph_0}, \supset^+, \neg^+, \{T^{\aleph^0}\}>$, where $\supset^+$ and $\neg^+$ are Boolean component-by-component operations. We select from $B^{\aleph_0}$ only those T-F-sequences (Boolean vectors) which have either a finite number of occurrences of T (possibly this number is 0) or a finite number of occurrences of F. The set of all such T-F-sequences $\alpha, \beta, \gamma, \ldots$ will be denoted by $Fin(\omega)$. In this case α^T and α^F will indicate that the number of occurrences of T or F is finite (or equal to 0).

For each $\alpha \in Fin(\omega)$ let $\eta(\alpha)$ be a finite number of occurrences of T or F in α such that:

$$\eta(\alpha) = \begin{cases} m & \text{if } \alpha \text{ is } \alpha^T \\ -m & \text{if } \alpha \text{ is } \alpha^F, \end{cases}$$

where $m, -m \in Z$ (Z is the set of integers). Then $\alpha \simeq \beta$ if $\eta(\alpha) = \eta(\beta)$ and $Fin(\omega)/\simeq$ is the factor-set of $Fin(\omega)$ by relation $\simeq$. The factor-set $Fin(\omega/\simeq$ will be supplied by operations $\neg\neg$ and $\rightarrow$ as follows: for $|\alpha|, |\beta| \in Fin(\omega)/\simeq$ let $\neg\neg|\alpha| = |\neg^{+}\alpha|$ and $|\alpha| \rightarrow |\beta| = |\alpha' \supset^{+} \beta'|$, where $\alpha' \in |\alpha|, \beta' \in |\beta|$ and $\alpha' R \beta'$. The relation R being defined as follows:

$$<a_1, \ldots, a_\omega> R <b_1, \ldots, b_\omega> \Leftrightarrow$$

(1) $\eta(\alpha^T) \leq \eta(\beta^T)$ or α is α^T and β is β^F, $\forall i < \omega\ (a_i = T \Rightarrow b_i = T)$.
(2) $\eta(\alpha^F) \leq \eta(\beta^F), \forall i < \omega\ (b_i = F \Rightarrow a_i = F)$.
(3) $\eta(\alpha^T) > \eta(\beta^T)$ or α is α^T and β is β^T, $\forall i < \omega\ (b_i = T \Rightarrow a_i = T)$.
(4) $\eta(\alpha^F) > \eta(\beta^F), \forall i < \omega\ (a_i = F \Rightarrow b_i = F)$.

PROPOSITION 3. *$\underline{M}_{Fin(\omega)/\simeq} = <Fin(\omega)/\simeq, \rightarrow, \neg\neg, \{|T^{\aleph_0}|\}>$ and $\underline{M}_{\Sigma} = <\Sigma, \mapsto, -, \{0^-\}>$ are isomorphic matrices [8].*

4. Algebraic structure of the elements of $Fin(\omega)/\simeq$

Let $\alpha_i, \alpha_j \in |\alpha^T|$, where $|\alpha^T| \in Fin(\omega)/\simeq$. For any two elements from $|\alpha^T|$ we introduce the lexicographic order $\alpha_i < \alpha_j$. Let $T < F$ and $\alpha_i = (a_1, \ldots, a_n, \ldots)$ and $\alpha_j = (b_1, \ldots, b_n, \ldots)$. Then $(a_1, \ldots, a_n, \ldots) < (b_1, \ldots, b_n, \ldots)$ means that for some $k, a_k < b_k$ and $a_m = b_m$, for all $m < k$. It is known that the and $a_m = b_m$, for all $m < k$. It is known that the lexicographic order is the linear order. Hence each $|\alpha^T|$ is a lexicographic LH-algebra with the first element α_1 and the last element (the greatest) α_ω.

No define dually the lexicographic order $<$ for the elements $\alpha_i, \alpha_j \in |\alpha^F|$, where $|\alpha^F| \in Fin(\omega)/\simeq$. In this case $F < T$ and the first element α_1 is the greatest. Introducing dually the operations on the elements from $|\alpha^F|$ we get the lexicographic algebras which are dual to LH-algebras i.e., we have Brouwer lexicographic algebras (LB-algebras).

Finally, we come to the conclusion that union of countable sets of lexicographic LH- and LB-algebras is the truth-value set of Łukasiewicz's infinite valued logic $Ł_\omega$.

5. $Ł_\omega$ and $\underline{M}_{Fin(\omega)/\simeq}$

From the proof of the correctness of the discrete model for $Ł_\omega$ [12] it is easy to get the following result:

THEOREM. $\underline{M}_{Fin(\omega)/\simeq}$ *is a factor-matrix for* $Ł_\omega$.

SKETCH OF THE PROOF. It was proved in [12] that if we take $\underline{M}_\Sigma$ as the truth-value set in case of the definition of an evaluation in possible worlds semantics with the ternary accessibility relation for $Ł_\omega$ then such a model is correct for $Ł_\omega$. It is easy to show that $Fin(\omega)/\simeq$ is an MV-algebra (using the definitions $|\alpha| + |\beta| = \neg\neg|\alpha| \to |\beta|$ and $|\alpha| \cdot |\beta| = \neg\neg(|\alpha| \to \neg\neg|\beta|)$. Then by [12], p. 62, we can associate with $\underline{M}_{Fin(\omega)/\simeq}$ some L-frame $< 0, 1, K, R, * >$ where K is the set of possible worlds, $0, 1 \in K$, R is a ternary accessibility relation on K and $*$ is the unary operation on K as in the case of the system R of entailment [11].

Defining $\underline{M}_\Sigma$ as a truth-value set for an evaluation on such an L-frame we get the correct model for $Ł_\omega$.

6. The matrix $\underline{M}_{\mathbf{A}} = <\mathbf{A}, \overset{Ł}{\to}, \overset{Ł}{\sim}, \{1\}>$

Let $\mathbf{A}$ be the union of countable sets of abstract LH- and LB-algebras. Note that the set $\mathbf{A}$ can be linearly-ordered by means of the category-theoretic tools, e.g., as in [1] taking the set of the elements of $\mathbf{A}$ as the set of objects of such a category. In that case the degenerate Heyting algebra $\mathbf{0}$ and the degenerate Brouwer algebra $\mathbf{1}$ will be the subobject and the factor-object of this category respectively. By usual way we can define the operations $\Rightarrow$ and $\dot{-}$ on the elements of the set $\mathbf{A}$.

Let us consider the algebra $H\text{-}B\text{-}M = < \mathbf{A}, \mathbf{0}, \mathbf{1}, \Rightarrow, \dot{-}, \oplus, \overset{Ł}{\sim}>$, where $\oplus$ is a monoid operation, and the operation $\overset{Ł}{\sim}$ (negation) transforms LH-algebra into LB-algebra and vice versa. Then

$$x \overset{Ł}{\to} y = (x \Rightarrow y) \oplus \overset{Ł}{\sim} (x \dot{-} y).$$

Let $\underline{M}_{\mathbf{A}} = < \mathbf{A}, \overset{Ł}{\to}, \overset{Ł}{\sim}, \{\mathbf{1}\} >$ be a logical matrix. An example of matrix $\underline{M}_{\mathbf{A}} = <\mathbf{A}, \overset{Ł}{\to}, \overset{Ł}{\sim}, \{\mathbf{1}\}>$ is the factor-matrix $\underline{M}_{A=\Sigma} = <\Sigma, \dot{\to}, \sim, \{0^-\}>$.

7. Conclusion

conclude that a logic has two algebraic levels. The first level (the inner one) consists of algebraic structures of truth-values of a given logic. For Łukasiewicz's infinite-valued logic $Ł_\omega$ such structures are *LH*- and *LB*-algebras. In general case various algebras can act as truth-values (even for the same logical system) and then there arises a complicated problem of constructing a logical theory permitting to ascribe various algebraic structures as truth-values to propositions.

Operations on algebras determine the second (external) level of a logic i.e., the algebra of the logic itself. In our case the algebra of $Ł_\omega$ (in the general case) is *H*-*B*-*M*-*S*-algebra. This algebra differs essentially from *MV*-algebras in which the operation $+$ and $\cdot$ are not lattice operations. The two mentioned algebraic levels of a logic directly lead to a category treatment of a logic and the constructing of a topos for it (see the article by Vladimir L. Vasyukov, the present issue of the Bulletin).

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