Our approach to a construction of semantics for modal calculi consists in interpretation of modal statement in terms of necessary and possible truths as well as in terms of necessary and possible falsity. It supposes also some actual (ontological) and logical necessity, chance and possibility.

In this paper there are given semantics for a number of calculi of actual and logical modalities. We have constructed the calculi of actual modalities in the beginning of the seventies.

Classical sentential logic (CSL) which is the base of many other calculi contains connectives “¬” (negation) and “⊃” (implication). The remaining connectives are introduced on the base of adequate definitions. As the symbols for actual necessity and possibility there are, respectively, used □ and ◇.

The calculus $S_{min}$ is the axiomatic strengthening of sentential calculus with axiom schemas (CSL) with the following axioms:

$AM_1$. □$A ⊃ A$;

$AM_2$. $A ⊃ ◇A$.

Semantics. To the semantics of CSL we added:

<table>
<thead>
<tr>
<th>□$A$</th>
<th>$t$</th>
<th>$f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>$f$</td>
<td>$f$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>◇$A$</th>
<th>$t$</th>
<th>$f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>$f$</td>
<td>$f$</td>
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</table>

The value $\frac{t}{f}$ means vaqueness: “it is either true or false”. If a subformula of a formula has the value $\frac{t}{f}$, we have to consider both cases: the subformula has the value $t$ or it has the value $f$. For the more detailed study of
functional values, see [1], [2]. The distinguished value is \textit{t}. For this system and all which follow the completeness metatheorems are proved.

The calculus $S_\pi$ is the strengthening of CSL with axioms:

\begin{align*}
AM_1. \quad & \square A \supset A, \\
AM_2. \quad & A \supset \Diamond A, \\
AM_3. \quad & \neg \square \neg A \supset \Diamond A, \\
AM_4. \quad & \Diamond A \supset \neg \square \neg A, \\
AM_5. \quad & \neg \Diamond A \supset \Box (A \supset B), \\
AM_6. \quad & \Box B \supset \Box (A \supset B), \\
AM_7. \quad & \Box (A \supset B) \supset (\Box A \supset \Box B), \\
AM_8. \quad & \Box (A \supset B) \supset (\Diamond A \supset \Box B), \\
AM_9. \quad & \Diamond (A \supset B) \supset (\Box A \supset \Diamond B), \\
AM_{10}. \quad & \Diamond B \supset \Diamond (A \supset B), \\
AM_{11}. \quad & \neg A \supset \Diamond (A \supset B)
\end{align*}

and the rule of inference that says we can replace any number of occurrences of $\neg \neg A$ by $A$ and vice versa.

Se m a n t i c s. Variables take their values from the domain \{\textit{tn}, \textit{tc}, \textit{fi}, \textit{fc}\}. The symbols \textit{tn}, \textit{tc}, \textit{fi}, \textit{fc} mean, respectively: “necessary truth”, “possible truth”, “necessary falsity”, “possible falsity”, \textit{t} and \textit{f} are understood as strigthenings of \textit{tn} and \textit{fc}, respectively. Let us define:

\begin{table}
\begin{tabular}{c|cccc}
\hline
\textit{A} & \textit{tn} & \textit{tc} & \textit{fi} & \textit{fc} \\
\hline
\textit{\square A} & \textit{t} & \textit{f} & \textit{f} & \textit{f} \\
\textit{\Diamond A} & \textit{t} & \textit{t} & \textit{f} & \textit{t} \\
\textit{\neg A} & \textit{f} & \textit{f} & \textit{t} & \textit{t} \\
\hline
\end{tabular}
\end{table}
The distinguished values are $t^n$ and $t^c$.

**Calculus $S_a$** we obtain $S_a$ by replacing the axiom schema $AM_8$ by the following two schemas:

$AM_8'$: $(A \supset B) \supset (\Diamond A \supset (\Diamond \neg B \supset (\neg A \supset \neg B)))$,

$AM_8''$: $(A \supset B) \supset (\Diamond A \supset \Diamond B)$.

**Semantics**. We accept all definitions for semantics for $S_a$ but the implication in which we change the following two cases:

$$t^c \supset t^c = \frac{t^n}{t^c},$$

$$f^c \supset f^c = \frac{t^n}{t^c}.$$ 

**Calculus $S_a+$** we obtain from $S_a$ by deleting the axiom $AM_8$.

**Semantics** differs from the one for $S_a$ in one case only: in the definition of implication we put:

$$f^c \supset t^c = \frac{t^n}{t^c}.$$ 

**Calculus $S_\varphi$** we obtain from $S_a$ by adding the following six axioms:

$AM_{12}$: $\Box A \supset \Box \Box A$,  

$AM_{13}$: $\Box \Box A \supset \Diamond A$,  

$AM_{14}$: $\Diamond A \supset \Diamond \Box A$,  

$AM_{15}$: $\Box A \supset \Box \Diamond A$,  

$AM_{16}$: $\Box \Diamond A \supset \Box A$,  

$AM_{17}$: $\Diamond \Diamond A \supset \Diamond A$.

**Semantics** differs from the one for $S_a$ in the definitions of $\Box$ and $\Diamond$:

<table>
<thead>
<tr>
<th>$A$</th>
<th>$t^n$</th>
<th>$t^c$</th>
<th>$f^i$</th>
<th>$f^c$</th>
<th>$f^i$</th>
<th>$f^c$</th>
<th>$f^i$</th>
<th>$f^c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Box A$</td>
<td>$t^n$</td>
<td>$t^c$</td>
<td>$f^i$</td>
<td>$f^c$</td>
<td>$f^i$</td>
<td>$f^c$</td>
<td>$f^i$</td>
<td>$f^c$</td>
</tr>
</tbody>
</table>
A Semantics for Modal Calculi

Calculus $S_\delta$ is obtained from $S_\delta$ by replacing axiom $AM_8$ by the following:

$AM_8'$. $\Box (A \supset B) \supset (\Box A \supset (\Box \neg B \supset (\neg A \supset \neg B)))$,

$AM_8''$. $\Box (A \supset B) \supset (\Box A \supset \Box B)$.

Semantics differs from the one for $S_\delta$ in two cases in the definition of implication:

$t^c \supset t^c = \frac{t^n}{t^c}$

$f^c \supset f^c = \frac{t^n}{t^f}$

Calculus $S_{\delta}^+$ is obtained from $S_\delta$ by deleting the axiom $AM_8'$.

Semantics differs from the one for $S_\delta$ in one case of the definition of implication:

$f^c \supset f^c = \frac{t^n}{t^f}$.

Calculus $S_\pi$ is obtained from $S_\pi$ by adding the following axioms:

$AM_{12}$. $\Box A \supset \Box \Box A$,

$AM_{13}$. $\Box \Box A \supset \Box A$,

$AM_{14}$. $\Box A \supset \Box \Box A$,

$AM_{15}$. $\Box \Box A \supset \Box A$.

Semantics differs from the one for $S_\pi$ in the definitions of $\Box$ and $\Box$:

$A \quad t^n \quad t^c \quad f^i \quad f^c$\hfill $\Box A \quad t^n \quad t^c \quad f^i \quad f^c$.

$A \quad t^n \quad t^c \quad f^i \quad f^c$\hfill $\Diamond A \quad t^n \quad t^c \quad f^i \quad t^n$.

Calculus $S_\pi$ is obtained from $S_\pi$ by replacing the axiom scheme $AM_8$ by the following two axioms:
AMs. \( \Box(A \supset B) \supset (\Diamond A \supset (\Diamond \neg B \supset (\neg A \supset \neg B))) \),
AMs’. \( \Box(A \supset B) \supset (\Diamond A \supset B) \).

Semantics differs from the one for \( S^c \) in two cases in the definition of implication:

\[
t^c \supset t^c = \frac{t^n}{t^c}
\]

\[
f^c \supset f^c = \frac{t^n}{t^c}
\]

Calculus \( S_{b+} \) is obtained from \( S_b \) by deleting the axiom scheme \( AMs' \).

Semantics differs from the one for \( S^c \) in one case in the definition of implication:

\[
f^c \supset t^c = \frac{t^n}{t^c}
\]

Calculus \( S_{c} \). Logical expressions: \( \neg, \Box, \Diamond, \land, \lor, \supset \). The definition of a formula is standard. Among formulas we distinguish modal ones:

a) if \( A \) is a formula, then \( \Box A \) and \( \Diamond A \) are modal formulas,
b) if \( A \) and \( B \) are modal formulas then \( \neg A, \Box A, \Diamond A, (A \land B), (A \lor B),
(A \supset B) \) are modal formulas,
c) nothing except the formulas described above is a modal formula.

The calculus \( S_c \) contains axiom schemata coinciding with the axiom schemata of CSL. The schemata include also metasigns \( A, B, C \) for modal formulas.

The remaining axioms are the following (\( A, B \) are any formulas):

\[
A_1. \Box A \supset A,
A_2. A \supset \Diamond A,
A_3. \neg \Box \neg A \supset \Diamond A,
A_4. \Diamond A \supset \neg \Box \neg A,
A_5. \Box A \supset \Box \Box A,
A_6. \Diamond \Box A \supset \Box A,
A_7. \Diamond A \supset \Box \Diamond A,
A_8. \Diamond \Diamond A \supset \Diamond A,
A_9. \Box A \land \Box B \supset (\Box A \land \Box B),
A_{10}. \Diamond A \land \Box B \supset (\Diamond A \land B),
\]
\[ \begin{align*}
A_{11}. \quad & \Box (A \land B) \supset \Box A \land \Box B, \\
A_{12}. \quad & \Box (A \supset B) \supset (\Box A \supset \Box B), \\
A_{13}. \quad & \Diamond (A \land B) \supset \Diamond A \supset \Diamond B, \\
A_{14}. \quad & \Box A \lor \Box B \supset \Box (A \lor B), \\
A_{15}. \quad & \Diamond A \lor \Diamond B \supset \Diamond (A \lor B), \\
A_{16}. \quad & \Box (A \lor B) \supset \Box A \lor \Box B, \\
A_{17}. \quad & \Diamond (A \lor B) \supset \Diamond A \lor \Diamond B, \\
A_{18}. \quad & \Diamond (A \lor B) \supset \Diamond A \lor \Diamond B, \\
A_{19}. \quad & \neg \Box A \supset \Box (A \supset B), \\
A_{20}. \quad & \Box B \supset \Box (A \supset B), \\
A_{21}. \quad & \Diamond B \supset \Diamond (A \supset B), \\
A_{22}. \quad & \Diamond \neg A \supset \Diamond (A \supset B), \\
A_{23}. \quad & \Box (A \supset B) \supset (\Diamond A \supset \Diamond B), \\
A_{24}. \quad & \Diamond (A \supset B) \supset (\Box A \supset \Box B).
\end{align*} \]

Rules of inference: Modus Ponens and the replacing of any number of occurrences of \( \neg A \) by \( A \) and vice versa.

**Semantics.** Variables take their values from a domain \( \{n, c, i\} \).

Definitions:

\[
\begin{align*}
\neg A & \\
\Box A & \\
\Diamond A & \\
A \supset B & \\
A \land B &
\end{align*}
\]
\[ \begin{array}{c|cccccccc} A & n & n & n & c & c & c & i & i \\ \hline B & n & c & i & n & c & i & n & c & i \\ A \lor B & n & n & n & n & n & \frac{2}{c} & c & n & c & i \end{array} \]

The distinguished value is \( n \). \( n, c, i \) mean, respectively: “necessary”, “possible”, “impossible”.

Essentially incomplete in respect to the axioms of \( S_\epsilon \) is the calculus \( S_\pi \), to which corresponds a semantics different from the one for \( S_\epsilon \) in one case of the definition of implication:

\[ c \supset c = c, \]

in one case of the definition of conjunction:

\[ c \land c = c \]

and in one case of the definition of disjunction:

\[ c \lor c = c. \]

(Note that the logic differs from the three-valued Lukasiewicz logic only in one case of the definition of implication:

\[ c \supset c = n, \]

if the meanings of 1, 1/2, 0 are understood as ours \( n, c, i \), respectively.)

**Lewis’ calculus \( S_5 \).** \( L, M \) are symbols for logical necessity and possibility, respectively. If a formula \( A \), in some state of affairs, takes the value \( f \), then clearly the formula \( LA \) takes in this state of affairs the value \( f \) as well. How to solve the problem about the \( LA \) truth-value in the case when \( A \) takes the value \( t \) ? Surely we should consider possible replacement of variables of \( A \) by sentences of different forms.

Let us denote by \( W \) the set of all description states for a formula. We introduce the notion of restricted set of description states for a formula (\( omoc \)). \( Omoc \) – it is an ordered set of two elements \( < Or, W' > \), where the restriction \( Or \) is a set of possible logical forms of logical formulas of elementary sentences, and \( W' \) is the set of description states obtained from \( W \) by restriction. The set of all \( omoc \)'s corresponds to the set of possible
substitutions in the place of propositional variables in formulas of various logical forms. For the full account of possible logical contents of elementary sentences we shall introduce the notion of subomoc of an omoc. If \( Or \) contains less than two formulas of the form \( Ca \), then the omoc itself is its subomoc. If \( Or \) contains more than one formula, for example it contains formulas \( Ca_i \) and \( Ca_e \), then on the base of a given omoc we construct subomoc’s by adding to \( Or \) formulas \( M(\tilde{a}_i \land \tilde{a}_e) \) or formulas \( \neg M(\tilde{a}_i \land \tilde{a}_e) \), where \( \tilde{a}_n \) is \( a_n \) or \( \neg a_n \) and similarly for three and more “occasional” variables. The restriction obtained by the way of enlargement of \( Or \) we shall denote by \( Or' \). In addition, \( W' \) may, as a result of enlargement of \( Or \), be restricted as well. Such a restricted set of description states we shall denote by \( W'' \). (\( W'' \subseteq W' \)). Clearly, in a description state \( \alpha (\alpha \in W'') \) of any subomoc \( \langle Or', W'' \rangle \), the values of formulas containing neither \( L \) nor \( M \) are defined. A formula \( LB \) takes the value \( t \) in some \( \alpha (\alpha \in W'') \) of a subomoc iff the formula \( B \) takes the value \( t \) in every \( \alpha \) of that subomoc. The formula \( MB \) takes the value \( t \) in some \( \alpha (\alpha \in W'') \) of a subomoc iff there exists such \( \beta (\beta \in W'') \), that \( B \) has the value \( t \) in \( \beta \). A formula is a tautology in a subomoc iff it has the value \( t \) in every \( \alpha (\alpha \in W'') \) of the subomoc. A formula \( A \) is a logical tautology iff it is a tautology in every subomoc.

References