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**T-F-TOPOSES FOR ŁUKASIEWICZ’S INFINITE-VALUED LOGICS**

The interpretation of the Łukasiewicz’s $\mathcal{L}_{\omega}$-infinite-valued logic $\mathcal{L}_{\omega}$ in toposes is proposed. The construction of the T-F-topos of functors $\text{Set}^{\Sigma}$ is defined by means of infinite T-F-sequences and then it is used for the interpretation of $\mathcal{L}_{\omega}$ in this topos. The equivalency of factor-semantics and T-F-toposes semantics for $\mathcal{L}_{\omega}$ is proved.

1. It is known that the category $\text{Set}^{\Sigma}$ of the functors for an arbitrary category $\mathcal{C}$ is a topos [1]. This fact was used in semantics only for intuitionistic logics. The present paper provides such a semantics for the system $\mathcal{L}_{\omega}$ of Łukasiewicz. In the role of $\mathcal{C}$ we shall use the Karpenko’s matrix $M_{\Sigma}$ (for details see [2]) which categorically turns out to be a finitely complete category of order. There are, as we know, connections between the set $\Sigma$ of $M_{\Sigma}$ and the set of the equivalency classes of the infinite T-F-sequences where every matrix element corresponds to the number of occurrences of $T$ (or $F$, respectively). In this sense we consider factor-semantics interpretation (see [2]) $\Sigma^+ \models x A$ (here $\Sigma^+ = \text{Fin}(\omega)/\sim$) where the truth-value of $A$ is defined by any T-F-sequence $\alpha$ with $\eta(\alpha) = x$. We get $\Sigma^+ \models A$ in case of all sequences that leads, in particular, to $\Sigma^+ \models \text{Fin}(\omega)/\sim$ where $\{\text{Fin}(\omega)/\sim\}$ plays the role of the set of sequences. Hereafter $\Box$ will mean the end of proof.

2. In the topos $\text{Set}^{\Sigma}$ the functor $\Omega : \Sigma \to \text{Set}$ must assign to each $p \in \Sigma$ the set of T-F-sequences $\{\alpha_i\}$ with $\eta(\alpha_i) = p$. But the set of sequences with the same $\eta(\alpha_i)$ represents either lexicographic Heyting algebra or lexicographic Brouwer one (see [2]) i.e., $\Omega(p) = \Omega p = \{\alpha : \eta(\alpha) = p\} = LH_p$ (or $LB_p$). Then $\Omega_{pq} : \Omega_p \to \Omega_q$ assigns to each $LH_p$ (or $LB_p$) a respective $LH_q$ (or $LB_q$) such that the occurrences of $T$ (or $F$) in $\beta \in LH_q$ (or $LB_q$) differs from the respective ones in $\alpha \in LH_p$ (or
LB_p) in \( \eta(\beta) - \eta(\alpha) \) cases. Moreover, we choose only the smallest (under the lexicographic order) elements of the set of such T-F-sequences. If the sequences belong to different types of algebras then we use the linearly ordered set \( \{LH\} \cup \{LB\} \), where \( LH < LB \). In order to get the concrete sequences it is enough to replace each \( T \) by \( F \) (or conversely) in sequences of one type and then to compare them. Using \( LHB \) to denote \( \{LH\} \cup \{LB\} \), by \( LHB_p \) we shall mean the respective Heyting or Brouwer algebra of the sequences with \( \eta(\alpha) = p \) for all \( \alpha \in LHB_p \). One easily shows that the lattice operations on algebras can be introduced according to their definitions on the elements.

The final object of the category \( \text{Set}^\Sigma \) is a constant functor \( 1 : \Sigma \to \text{Set} \) which is determined by the condition \( 1_p = \{ |T^{\aleph_0}| \} \) for \( p \in \Sigma \) and \( 1_{pq} = id_{\{ |T^{\aleph_0}| \}} \) under \( p \leq q \). The subobject classifier \( \text{true} : 1 \to \Omega \) is a natural transformation, for which \( p \)-th component is defined by the equality \( \text{true}(|T^{\aleph_0}|) = LHB_p \) i.e., the function \( \text{true} \) chooses the respective greatest lexicographic algebra.

If \( \tau : F \to G \) is an arbitrary subobject of \( \text{Set}^\Sigma \)-object \( G \) then each component \( \tau_p \) is injective and we can regard it as an inclusion function \( F_p \hookrightarrow G_p \). \( p \)-th component of the characteristic arrow \( \chi_\tau : G \to \Omega \) is defined as follows:

\[
(\chi_\tau)_p(x) = \text{the lexicographically smallest } T-F\text{-sequence } \alpha \in LHB_p, \text{ for which } G_{pq}(x) = F_q i.e., \text{ when all occurrences of } T (\text{or } F) \text{ in } \alpha \text{ are the same as in case of } F_p \text{ (here } x \in G_p).\]

The commutativity of the diagram

\[
\begin{array}{ccc}
G_p & \xrightarrow{(\chi_\tau)_p} & \Omega_p \\
\downarrow & & \downarrow \\
G_q & \xrightarrow{(\chi_\tau)_q} & \Omega_q \\
\end{array}
\]

under \( p \leq q \) is equivalent to the assertion that \( \chi_\tau \) is a natural transformation of functor \( G \) to the functor \( \Omega \). To show \( \Omega \)-axiom holds in our case, we take the sequence of the commuting diagrams

\[
\begin{array}{ccc}
F_p & \xrightarrow{\tau_p} & G_p \\
\downarrow & \Rightarrow & \downarrow \\
F_q & \xrightarrow{\tau_q} & G_q \\
\end{array} \quad \Rightarrow \quad \begin{array}{ccc}
F_p & \xrightarrow{\tau_p} & G_p \\
\downarrow & \Rightarrow & \downarrow \\
\{ |T^{\aleph_0}| \} & \xrightarrow{\text{true}} & \Omega_p \\
\end{array} \quad \Rightarrow \quad \begin{array}{ccc}
F & \xrightarrow{\tau} & G \\
\downarrow & \Rightarrow & \downarrow \\
\Omega & \xrightarrow{\chi_\tau} & \Omega \\
\end{array}
\]
where two first ones are the pullbacks. Then we consider the reverse sequence of the commuting diagrams

\[
\begin{array}{cccc}
\bullet & \xrightarrow{\sigma} & \Omega & \xleftarrow{\Omega_p} \\
F & \xrightarrow{\tau} & G & \xleftarrow{G_p} \\
1 & \xrightarrow{\text{true}} & \Omega & \xleftarrow{\Omega_q} \\
\end{array}
\]

where two first ones are the pullbacks. From the second pullback we have

\((*)\) \(F_q = \{x : \sigma(x) \in LH_B_q\}\).

Then \(\alpha = (\chi_r)_p(x) \iff G_{pq}(x) \iff \sigma_q(G_{pq}(x)) = \beta \in LH_B_q\) (by \((*)\)) \(\iff \Omega_{pq}(\sigma_p(x)) = \beta \in LH_B_p\) (due to the last diagram) \(\iff \sigma_p(x) \land \beta = \alpha \in LH_B_p\) (by the definition of \(\Omega_{pq}\)) \(\iff \alpha = \sigma_p(x)\). Thus \((\chi_r)_p(x) = \sigma_p(x)\). Since an arbitrary \(p \in \Sigma\) and \(x \in G_p\) has been used, we have \(\sigma = \chi_r\) and \(\Omega\)-axiom holds.

The initial object \(0 : \Sigma \to \text{Set}\) in the category \(\text{Set}^\Sigma\) will be a constant functor such that for \(p \leq q\) we have \(0_p = \{\{F^{R_0}\}\}\) and \(0_{pq} = id_{\{\{F^{R_0}\}\}}\). The components of the natural transformation \(0 \to 1\) for any \(p\) are the maps \(\{\{F^{R_0}\}\} \to \{\{T^{R_0}\}\}\). An arrow \(\text{false}\) is a characteristic arrow of the subobject \(i : 0 \to 1\) by the definition. For its component \(\text{false}_p : \{\{T^{R_0}\}\} \to \Omega_p\) we have \(\text{false}_p(\{\{T^{R_0}\}\}) = \{\{F^{R_0}\}\}\).

Now we can define a negation as an arrow \(\neg : \Omega \to \Omega\) being a characteristic arrow of the subobject \(\text{false}\). Then \(p\)-th component \(\neg_p : \Omega_p \to \Omega_p\) for any \(\alpha \in LH_p\) gives us \(\neg_p(\alpha) = \beta \in LB_p\) i.e., we substitute all the occurrences of \(T\) by \(F\) and conversely for \(\alpha \in LB_p\). Denoting this operation as \(*\) we get \(LH^*_p = LB_p\) and \(LB^*_p = LH_p\).

Let us consider the functor \(\mathcal{B} : \Sigma \to \text{Set}\). Its \(p\)-th component can be defined as \(\mathcal{B}_p = \{< LH_B_q, LH_B_r> : \alpha R_\beta \& \alpha \in LH_B_q \& \beta \in LH_B_r \& p = q + r \} \subseteq \Omega_q \times \Omega_r\), where \(R\) is a tolerantness relation from [2]. For a given functor \(\mathcal{B}\) we define a coequalizer \(e : \mathcal{B} \Rightarrow \Omega \times \Omega\) of the two arrows with the respective components \(LH_B_q, LH_B_r\) by using the natural transformation \(e\) which components are the inclusions \(e_p : \mathcal{B}_p \hookrightarrow \Omega_q \times \Omega_r\). Now we can introduce an implication \(\not\to : \Omega \times \Omega \to \Omega\) being the characteristic arrow of \(e\). Its \(p\)-th component is \(\not\to_p (< LH_B_q, LH_B_r>) = LH_B_q^{L_p}\)
3. The connection between $\text{Set}^\Sigma$ and $\text{M}_\Sigma$-validity is established by the following theorem.

**Theorem 1.** $\text{Set}^\Sigma \models A \iff \Sigma^+ \models A$.

Let $v : \varphi \rightarrow \Sigma^+$, where $\varphi$ – the set of the formulas of $L_{\aleph_0}$. Using the valuation $v$ we can define $\text{Set}^\Sigma$-valuation $v' : \varphi \rightarrow \text{Set}^\Sigma(1, \Omega)$. The function $v'$ assigns to every propositional letter its truth value $v'(\Pi) : 1 \rightarrow \Omega$ in $\text{Set}^\Sigma$.

The component $v'(\Pi)_p : \{ [T^\aleph_0] \} \rightarrow \Omega_p$ one can get by

\[ (**) \quad v'(\Pi)_p ([T^\aleph_0]) = v(\Pi) \cap LHB_p = v(\Pi)_p. \]

Thus $v'(\Pi)_p$ collects all the sequences with $\eta(\alpha) = p$ in which $\Pi$ is true.

Looking at the diagram

\[
\begin{array}{ccc}
\{ [T^\aleph_0] \} & \overset{v'(\Pi)_p}{\longrightarrow} & LHB_p \\
\downarrow & & \downarrow \\
\{ [T^\aleph_0] \} & \overset{v'(\Pi)_q}{\longrightarrow} & LHB_q \\
\end{array}
\]

(which obviously commutes), we conclude that $v'(\Pi)$ is a natural transformation.

**Lemma 1.** For any $A \in \varphi$, the $p$-th component $v'(A)_p : \{ [T^\aleph_0] \} \rightarrow LHB_p$ of the natural transformation $v'(A)$ satisfies the equality

\[ v'(A)_p ([T^\aleph_0]) = v(A)_p. \]

**Proof.** By the induction on the construction of a wff $A$. In case of $A = \Pi$ it follows from $(**)$. For $A = \neg B$ (when the lemma is proved for $B$) we have $v'(\neg B)_p = (\neg \circ v'(B))_p = \neg_p \circ v'(B)_p$, hence $v'(A)_p ([T^\aleph_0]) = \neg_p (v'(B)_p) ([T^\aleph_0]) = \neg_p (v(B)_p)$ (by the induction hypothesis) $\neq (v(B))_p$ (by the definition of the negative arrow) $\neq v(\neg B)_p$ (by the definition of the $\Sigma$-valuation) $\neq v(A)_p$. 

$LHB_r = \{ \alpha \supset \beta : \alpha \in LHB_q \& \beta \in LHB_r \& \alpha R \beta \}$, where $\alpha \supset \beta = < a_1, \ldots, a_n, \ldots > \supset < b_1, \ldots, b_n, \ldots > = < a_1 \oplus b_1, \ldots, a_n \oplus b_n, \ldots >$ and the implication $\supset \oplus$ is a Boolean implication on the set $\{ T, F \}$ (see [2]).
Theorem 2. \( \vdash A \Rightarrow \Sigma^+ \models A. \)

Corollary 2.

\[ \forall \alpha \in A \Rightarrow \Sigma^+ \models A. \]

Proof. If \( \Sigma^+ \models A \) then \( v'(A) = \text{true} \). Hence for any \( p \) we have \( v'(A)_p = \text{true}_p(\{|T^{[\alpha]}|\}) = LHB_p \). Since \( p \) is arbitrary, we get, in particular, \( v'(A)_{0^+} = \{|T^{[\alpha]}|\} \). Consequently, \( \Sigma^+ \models A \Rightarrow \Sigma^+ \models A. \)

The arrow \( v'(\Pi) : 1 \to \Omega \) chooses the respective \( LHB_p \) for every \( p \in \Sigma \). In such a case \( v(\Pi) \) can be defined as the union of all such \( LHB \)

\[ \text{i.e., } v(\Pi) = \bigcup \{ v'(\Pi)_p(\{|T^{[\alpha]}|\}) : p \in \Sigma \} , \quad \text{or} \]

\[ (\ast \ast \ast) \alpha \in v(\Pi) \iff \exists p \in \Sigma[\alpha \in v'(\Pi)_p(\{|T^{[\alpha]}|\})]. \]

Lemma 3. For any \( p \in \Sigma \), \( v(\Pi) \land LHB_p = v'(\Pi)_p(\{|T^{[\alpha]}|\}) \), where \( v(\Pi) \) is considered by \( (\ast \ast \ast) \).

Proof. From \( (\ast \ast \ast) \) obviously follows that \( v'(\Pi)_p(\{|T^{[\alpha]}|\}) \subseteq v(\Pi) \). Moreover, if \( v'(\Pi)_p(\{|T^{[\alpha]}|\}) \subseteq v(\Pi) \) for every \( p \in \Sigma \), then \( v(\Pi)_p(\{|T^{[\alpha]}|\}) \subseteq LHB_p \). Therefore, \( v'(\Pi)_p(\{|T^{[\alpha]}|\}) \subseteq v(\Pi) \land LHB_p \). Conversely, from \( v'(\Pi)_p(\{|T^{[\alpha]}|\}) \subseteq LHB_p \) we get \( v(\Pi) \land LHB_p \subseteq v'(\Pi)_p(\{|T^{[\alpha]}|\}). \)

Corollary 4. \( \Sigma^+ \models A \Rightarrow \Sigma^+ \models A . \)

Proof. From \( \Sigma^+ \models A \) for any \( p \in \Sigma \) we obtain \( v(A)_p = v(A) \land LHB_p = LHB_p = \text{true}_p(\{|T^{[\alpha]}|\}) \). By Lemma 1, \( v'(A)_p(\{|T^{[\alpha]}|\}) = \text{true}_p(\{|T^{[\alpha]}|\}) \), i.e., \( v'(A) = \text{true} \). "}

Corollaries 2 and 4 give us the proof of the Theorem 1. Finally, from \( \vdash L_{\Sigma^+} A \iff \Sigma \models A \Rightarrow \Sigma^+ \models A \) we get the proof of the following theorem:

Theorem 2. \( \vdash L_{\Sigma^+} A \iff \Sigma^+ \models A. \)
References
