A CHARACTERIZATION OF CONSEQUENCE OPERATIONS PRESERVING DEGREES OF TRUTH

Formalization of reasoning which accepts rules of inference leading to conclusions whose logical values are not smaller than the logical value of the "weakest" premise leads to the concept of consequence operation preserving degrees of truth. Several examples of such consequence operations have already been considered (see e.g. [5]). In the present paper we give a general notion of the consequence operation preserving degrees of truth and its characterization in terms of projective generation and selfextensionality.

1. Introduction

1.1. By an abstract logic we understand any pair \( L = (\mathcal{A}, C) \) or \( L = (\mathcal{A}, c) \), where \( \mathcal{A} \) is an abstract algebra, \( C \) is a closure system over \( \mathcal{A} \) and \( c \) – a closure operation on \( \mathcal{A} \). We identify the pairs \( (\mathcal{A}, C) \), \( (\mathcal{A}, c) \) when \( C \) and \( c \) are such that \( C = \text{Th}(c) \), where \( \text{Th}(c) = \{ X \subseteq \mathcal{A} : X = c(X) \} \).

Let \( \{ L_i : i \in I \} \) be a family of abstract logics such that for any \( i \in I : L_i = (\mathcal{A}_i, C_i) \), where \( \{ \mathcal{A}_i : i \in I \} \) is a family of similar algebras. Let \( \mathcal{A} \) be an algebra similar to \( \mathcal{A}_i, i \in I \) and \( H = \{ h_i : i \in I \} \), where for any \( i \in I \), \( h_i \in \text{Hom}(\mathcal{A}, \mathcal{A}_i) \), \( \text{Hom}(\mathcal{A}, \mathcal{A}_i) \) is the set of all homomorphisms \( h : \mathcal{A} \rightarrow \mathcal{A}_i \) and let \( C \) be the smallest closure system over \( \mathcal{A} \) containing the family: \( \bigcup \{ \{ h_i(X) : X \in C_i \} : i \in I \} \). Then we say (cf. [1]) that the abstract logic \( L = (\mathcal{A}, C) \) is projectively generated from the family of logics \( \{ L_i : i \in I \} \) by \( H \).

1.2. Where \( \mathcal{S} = (S, F_1, \ldots, F_n) \) is a propositional language, \( \mathcal{A} \) – an algebra similar to \( \mathcal{S} \), \( \mathcal{A} \) – a closure system over \( \mathcal{A} \) and \( c^\mathcal{A} \) – the closure
operation on $A$ such that $Th(c^A) = A$, we define (cf. [3]) the map $C^A : P(S) \rightarrow P(S)$ in the following way: for any $\alpha \in S, X \subseteq S : \alpha \in C^A(X)$ iff $\forall h \in Hom(S, A) : h(\alpha) \in c^A(h(X))$.

One may show that $C^A$ is a structural consequence operation on $S$ and the abstract logic $(S, C^A)$ is projectively generated from $\{(A_h, A_h) : h \in Hom(S, A)\}$ by $Hom(S, A)$, where for each $h \in Hom(S, A), A_h = A$ and $A_h = A$.

In what follows we shall also say that the consequence operation $C^A$ is projectively generated from the abstract logic $(A, [B])$.

1.3. Theorem (cf. [1]). Let $A$ be any algebra similar to the language $S$ and for any $B \subseteq P(A)$ let $[B]$ denote the smallest closure system over $A$ containing $B$. Then the consequence operation $C_n(A, B)$ determined on $S$ by the generalized matrix $(A, B)$ is projectively generated from the abstract logic $(A, [B])$.

Corollary. Let $A$ be any algebra similar to the language $S$.

(1) For any closure system $A$ over $A : C^A = \inf\{C_n(A, D) : D \subseteq A\}$ where for each $D \subseteq A : C_n(A, D)$ is the consequence operation determined by the matrix $(A, D)$ and $\inf$ is the operation of the greatest lower bound in the lattice of all consequence operations on $S$.

(2) For any $D \subseteq A : C_n(A, D) = C^A_D$, where $A_D = \{D, A\}$.

2. $PDTw$-property

2.1. Let $C$ be any consequence operation on the language $S$. We shall state that $C$ weakly preserves degrees of truth (or $C$ has the $PDTw$-property) if and only if there exist:

(1) a complete lattice $(A, \leq)$
(2) an algebra $R$ similar to $S$ with $R \subseteq A$ such that for any $\alpha \in S, X \subseteq S$ ($\ast$) $\alpha \in C(X)$ iff $\forall h \in Hom(S, R) : inf_A h(X) \leq h(\alpha)$.

We shall say that $C$ preserves degrees of truth (or $C$ has the $PDT$-property) iff $C$ has the $PDTw$-property and $R = A$. 
Lemma 2.2. Let $C$ be any consequence operation on the language $S$. $C$ has the PD$T^w$-property iff $C$ is projectively generated from an abstract logic $(R, \mathcal{F}^R)$, where $R$ is the algebra similar to $S$ such that $R \subseteq A$ for some complete lattice $(A, \leq)$ and $\mathcal{F}^R$ is the family of all subsets $F$ of the set $R$ such that:

$\forall x \in R \ (\inf_A (F \cup \{x\}) = \inf_A F \Rightarrow x \in F)$ i.e.

$\mathcal{F}^R = \{[a] \cap R : a \in A\}$, where $[a] = \{x \in A : a \leq x\}$.

Proof. Let $(A, \leq)$ be a complete lattice, $R$ - an algebra similar to $S$ such that $R \subseteq A$ and $\mathcal{F}$ - the closure system of all principal filters of the lattice $(A, \leq)$. Then for any $B \subseteq A$:

1. $c^F(B) = [\inf_A B]$, where $c^F$ is the closure operation on $A$ such that $Th(c^F) = \mathcal{F}$.

Given any $\alpha \in S, X \subseteq S, h \in \text{Hom}(S, R)$, from (1) we obtain:

2. $\inf_A h(X) \leq h(\alpha)$ iff $h(\alpha) \in c^F(h(X)) \cap R$.

But for any $B \subseteq R$:

3. $c^F(B) \cap R = c^{\mathcal{F}^R}(B)$, where $c^{\mathcal{F}^R}$ is the closure operation on $R$ such that $Th(c^{\mathcal{F}^R}) = \{[a] \cap R : a \in A\}$.

If $C$ has the PD$T^w$-property, i.e. there exist an algebra $R$ and a complete lattice $(A, \leq)$ such that the condition $(\ast)$ from 2.1 holds, then according to (2) and (3): $C = C^{\mathcal{F}^R}$, so in other words, $C$ is projectively generated from $(R, \mathcal{F}^R)$.

Conversely, if there exist a complete lattice $(A, \leq)$ and an algebra $R$ similar to $S$, $R \subseteq A$, such that $C = C^{\mathcal{F}^R}$, then due to (2) and (3), the condition $(\ast)$ holds. Hence $C$ has the PD$T^w$-property.

2.3. We recall (cf. [4]) that a structural consequence operation $C$ on the language $S$ is called selfextensional when its greatest logical equivalence $\Theta_C$, i.e. the relation on $S$ defined as follows: for each $\alpha, \beta \in S : \alpha \equiv \beta(\Theta_C)$ iff $C(\alpha) = C(\beta)$, is a congruence relation of the language $S$. 

Theorem. Let \( C \) be a consequence operation on the language \( S \). \( C \) has the PDT\( w \)-property iff \( C \) is selfextensional.

Proof. \((\Rightarrow)\) Assume that \( C \) weakly preserves degrees of truth. Let \( \Theta_C \) be the greatest logical equivalence of the consequence \( C \). Then for each \( \alpha, \beta \in S \):
\[
\alpha \equiv \beta (\Theta_C) \text{ iff } \forall h \in \text{Hom}(S, R) : h(\alpha) = h(\beta),
\]
where \( R \) is the algebra from 2.1. Hence \( \Theta_C = \bigcap \{ \Theta_h : h \in \text{Hom}(S, R) \} \), where for any \( \alpha, \beta \in S, h \in \text{Hom}(S, R) : \alpha \equiv \beta (\Theta_h) \text{ iff } h(\alpha) = h(\beta) \). For it is obvious that for each \( h \in \text{Hom}(S, R), \Theta_h \) is a congruence of \( S \) then the consequence \( C \) (as structural) is selfextensional.

\((\Leftarrow)\) Assume that \( C \) is selfextensional. So there exists (cf. [4]) a referential matrix \( (R, \{D_t : t \in T\}) \), where \( T \neq \emptyset, R \subseteq \{0,1\}^T \), for each \( t \in T : D_t = \{ r \in R : r(t) = 1 \} \), which is adequate for \( C \). Hence from Theorem 1.3 we obtain that \( C = C^{\{D_t : t \in T\}} \) i.e. \( C \) is projectively generated from the abstract logic \( (R, \{D_t : t \in T\}) \).

The partially ordered set of characteristic functions of all subsets of the set \( T : (\{0,1\}^T, \leq) \), where the relation \( \leq \) is defined as follows: for any \( r_1, r_2 \in \{0,1\}^T : r_1 \leq r_2 \text{ iff } \forall t \in T : r_1(t) \leq r_2(t) \), obviously forms a complete lattice. Moreover, for each \( t \in T \):

\[
(1) \ D_t = [r_t] \cap R, \text{ where } r_t \in \{0,1\}^T \text{ is defined as follows:}
\]
\[
\forall s \in T : r_t(s) = \begin{cases} 1 & \text{if } s = t \\ 0 & \text{otherwise.} \end{cases}
\]

One may show, due to (1), that:

\[
(2) \ \{D_t : t \in T\} = \{R \cap \nabla : \nabla \in \mathcal{F}\}, \text{ where } \mathcal{F} \text{ is the family of all principal filters of the lattice } (\{0,1\}^T, \leq).
\]

Consequently, from (2), due to Lemma 2.2 we obtain that \( C \) has the PDT\( w \)-property.

3. PDT-property

3.1. Lemma. Let \( C \) be a consequence operation on the language \( S \). \( C \) has the PDT\( t \)-property iff \( C \) is projectively generated from an abstract logic.
$(A, \mathcal{F})$, where $A$ is the algebra similar to $S$ such that there is a partial ordering $\leq$ on the carrier $A$ of the algebra $A$, and $(A, \leq)$ is a complete lattice. $\mathcal{F}$ is the family of all principal filters of $(A, \leq)$.

**Proof.** This Lemma is an obvious corollary to Lemma 2.2.

3.2. **Lemma.** Let $c$ be any closure operation on a set $A$ and $\Theta_c$ – its greatest logical equivalence. Consider the partially ordered set $(A/\Theta_c, \leq)$ with $\leq$ defined in the following way: for any $a, b \in A : [a]_{\Theta_c} \leq [b]_{\Theta_c}$ iff $b \in c(a)$. Then the following conditions are equivalent:

(i) $(A/\Theta_c, \leq)$ is a complete lattice
(ii) the family $\{c(a) : a \in A\}$ is a closure system over $A$.

**Proof.** It is obvious that (ii) is equivalent to

(1) $\forall X \subseteq A \exists b \in A : c(b) = \bigcap\{c(a) : a \in X\}$. 

One may show that for each $b \in A$ and $X \subseteq A : [b]_{\Theta_c} = \text{sup}_{\leq} \{[a]_{\Theta_c} : a \in X\}$ if and only if $c(b) = \bigcap\{c(a) : a \in X\}$. So (1) is equivalent to the following condition: for any $X \subseteq A$ (including $X = \emptyset$) for the family $\{[a]_{\Theta_c} : a \in X\}$ there exists $\text{sup}_{\leq} \{[a]_{\Theta_c} : a \in X\}$. Since every partially ordered set $(P, \leq)$ in which for any subset $R \subseteq P$ (including $R = \emptyset$) the least upper bound exists, forms a complete lattice, we conclude that the later condition is equivalent to (i).

Notice that for each $b \in A$ and $X \subseteq A : [b]_{\Theta_c} = \text{inf}_{\leq} \{[a]_{\Theta_c} : a \in X\}$ if and only if $c(b) = \bigcap\{c(y) : X \subseteq c(y), y \in A\}$.

3.3. Let $c$ be any closure operation on a set $A$. We say that $c$ is one-axiomatizable if for any theory $X \in \text{Th}(c)$ there is an element $a \in A$ such that $X = c(a)$.

**Theorem.** Let $C$ be any consequence operation on the language $S$. $C$ has the PDT-property iff $C$ is projective generated from an abstract logic $(A, c)$ such that $c$ is one-axiomatizable and its greatest logical equivalence $\Theta_c$ is a congruence of the algebra $A$. 

Proof. \((\Rightarrow)\) Assume that \(C\) has the PDT-property. Then due to Lemma 3.1, \(C\) is projectively generated from some abstract logic \((A, c^F)\). One may show that \(c^F\) is one-axiomatizable and that \(\Theta_{c^F}\) is the identity.

\((\Leftarrow)\) Let \((A, c)\) be an abstract logic such that \(Th(c) = \{c(a) : a \in A\}\) and \(\Theta_c\) be a congruence of the algebra \(A\) (\(A\) is similar to \(S\)). Assume that \(C\) is projectively generated on \(S\) from the logic \((A, c)\). Which, according to Corollary 1.3, yields: \(C = \inf \{C_n(A, c(a)) : a \in A\}\). Due to Lemma 3.2, \((A/\Theta_c, \leq_c)\) is a complete lattice. Furthemore, for any \(a \in A\) the relation \(\Theta_c\) is a congruence of the matrix \((A, c(a))\) and \(c(a)/\Theta_c = [a]_{\Theta_c, \leq_c}\), where \([a]_{\Theta_c, \leq_c}\) is the principal filter of the lattice \((A/\Theta_c, \leq_c)\) generated by the element \([a]_{\Theta_c}\). Thus \(C = \inf \{C_n(A/\Theta_c, [a]_{\Theta_c, \leq_c}) : a \in A\}\). Hence, due to Corollary 1.3 and Lemma 3.1, we conclude that \(C\) has the PDT-property.

3.4. The consequences which have PDT-property have already been considered. The well-known example is \(n\)-valued well-determined Łukasiewicz logic \(E(L_n)\) which is defined as follows: for any \(\alpha \in S, X \subseteq S: \alpha \in E(L_n)(X)\) iff \((\alpha_1 \land \ldots \land \alpha_n) \rightarrow \alpha \in E(L_n)\) for some \(\alpha_1, \ldots, \alpha_n \in X \cup E(L_n)\), where \(E(L_n)\) is the content of the \(n\)-valued Łukasiewicz matrix \(L_n = ([L_n], \{1\})\). One may show that for any \(\alpha \in S, X \subseteq S: \alpha \in E(L_n)(X)\) iff \(\forall h \in Hom(S, [L_n]) : \inf h(X) \leq h(\alpha)\), where "\(\inf\)" is the symbol of the operation of the greatest lower bound in the chain \(L_n = \{0, 1/(n-1), \ldots, (n-2)/(n-1), 1\}\), cf. [5], p. 67.

References

Department of Logic
Łódź University
ul. Matejki 34a
90-237 Łódź, Poland