LOWER LEVEL CONNECTIONS BETWEEN REPRESENTATIONS OF RELATION ALGEBRAS

(abstract)

The algebra of all binary relations on a given set is the most important example of a relation algebra (cf. [3]). In this note we will examine the possible isomorphisms within some subclasses of a closely related class (cf. [1] 5.3.2);

\(A\) is a relation set algebra with base \(U\) if its Boolean reduct is a field of sets with unit element \(2^U\), its universe \(A\) contains the identity relation on \(U\) and it is closed under the operations \(-1\) and \(|\), where \(R|S = \{<x,y> : <x,z>\in R <z,y>\in S\text{ for some }z\}\) and \(R^{-1} = \{<x,y> : <y,x>\in R\}\) for any \(R, S \subseteq 2^U\). The class of all relation set algebras is denoted by \(Rs\).

Considering an algebra with a universe which is defined as a collection of some subsets of a given set, the question naturally arises when and to what extent the algebraic structure of this algebra determines its set structure. There are, in fact, cases when the only possible isomorphisms are the trivial ones, the so-called base – isomorphisms (cf. [2] I.3.5-6 on p. 37).

Let \(A\) be a relation set algebra. We say that a function \(f\) defined on \(A\) is a base-isomorphism if there is a one-one function \(g\) defined on the base of \(A\) such that \(fR = \{<gx, gy> : <x,y>\in R\}\) for any \(R \in A\).

In order to be able to examine more general cases as well, we introduce some weaker kinds of isomorphisms between set algebras defined in terms of their set structure (cf. [2] 3.1 on pp. 155–156);

Let \(A\) be a relation set algebra with base \(U\). We say that an isomorphsim \(f\) from \(A\) onto a relation set algebra is an ext-isomorphism if there is a \(V \subseteq U\) such that \(fR = R \cap 2^V\) for any \(R \in A\). Further, \(f\) is an ext-base-isomorphism if \(f = e \circ b\) for some base-isomorphism \(b\) and ext-isomorphism \(e\), while \(f\) is said to be a lower-base-isomorphism if \(f = e^{-1} \circ b \circ e_2\) for some ext-isomorphisms \(e_1, e_2\) and base-isomorphism \(b\).
As to the conditions under which all isomorphisms must be one of those defined above, we will use a notion introduced by A. Tarski (cf. [8] Section 4.1):

Let $A$ be a relation algebra, $p, q \in A$ are conjugated quasiprojections if $p^{\prime}; p \leq 1, q^{\prime}; q \leq 1$ and $p^{\prime}; q = 1$. We denote the class of all relation set algebras having conjugated quasiprojections by $QRs$.

Finally, an algebra with a universe which is a subset of a power set is said to be compact if the intersection of any subset of its universe with the finite intersection property is not empty.

Now, we are ready to formulate our result. Note that every simple relation algebra with conjugated quasiprojections is isomorphic to a relation set algebra (cf. [4] and [8] Section 8.4).

Theorem 1.

(i) Let $A \in QRs$ be countable.

(a) If $A$ is atomic, then every isomorphism from $A$ onto a relation set algebra is a lower-base-isomorphism.

(b) If $A$ is compact, then every isomorphism from $A$ onto a relation set algebra is a lower-base-isomorphism.

(ii) Let $A \in QRs$ be countable.

(a) If the base of $A$ is countable, then $A$ is atomic iff every isomorphism from $A$ onto a relation set algebra is a lower-base-isomorphism.

(b) $A$ is compact iff every isomorphism from $A$ onto a relation set algebra with a countable base is an ext-base-isomorphism.

(iii) (a) If $A \in QRs$ is compact, then every isomorphism from $A$ onto a relation set algebra with a countable base is an ext-base-isomorphism.

(b) If $A, B \in QRs$ are isomorphic, then there is a compact relation set algebra which is ext-base-isomorphic to both $A$ and $B$.

The proof of the theorem above is based on a result which connects $QRs$ with the class of $\omega$-dimensional cylindric set algebras. In fact, Theorem 1 can be inferred from some cylindric set algebraic results of [5] and [6] using Theorem 2 below. (The results proved in [6] are reported in [7].) The notation and terminology used in the following definition and theorem are those of [1]. For the sake of simplicity we will not take into consideration the difference between the sets $X \subseteq ^2U$ and $\{x \in ^\omega U : <x_0, x_1> \in X\}$.
For any $A \in Rs$, let $C(A)$ be the subalgebra of the full $\omega$-dimensional cylindric set algebra with the same base as that of $A$ generated by $A$.

**Theorem 2.**

(i) $A = Nr_2C(A)$ for any $A \in QRs$.

(ii) Let $A, B \in QRs$.

$f \in Is(A, B)$ iff there is a $g \in Is(C(A), C(B))$ such that $A_1g = f$.

Moreover, $f$ is an ext (resp. base)-isomorphism iff so is $g$.

(iii) Let $A \in QRs$.

(a) $A$ is atomic iff so is $Nr_nC(A)$ for any $n \in \omega$,

(b) $A$ is compact iff so is $C(A)$.

**References**


Technical University of Budapest
Electrical Engineering Faculty
Department of Mathematics
1111, Stoczek u. 2., H ep., Hungary