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A HYPOTHESIS ON THE FINITENESS OP GRAPHS FOR ŁUKASIEWICZ'S PRECOMPLETE LOGICS (GRAPHS FOR PRIME NUMBERS)

The article offers on the basis of functional characteristics of Łukasiewicz's finite-valued logics a partition of a natural series of numbers into equivalence classes such that every class has one and only one prime number. This causes prime numbers to be represented as rooted trees. A hypothesis is suggested that such representation is finite for every prime number and necessary condition for it is specified.

1. Preliminaries

Let $M^{Ł}_{n+1} = < V, \sim, \rightarrow, \{1\} >$ where $n \in N$ and $n \geq 2$, be Łukasiewicz's $(n+1)$-valued matrix. That is, $V = \{0, 1/n, \ldots, n-1/n, 1\}, \sim x = 1 - x, x \rightarrow y = min(1, 1-x+y)$ and $\{1\}$ is the set of designated elements of $M^{Ł}_{n+1}$. The propositional Łukasiewicz's logic $Ł_{n+1}$ is defined as a set of all tautologies of the matrix $M^{Ł}_{n+1}$ [4].

We denote the set of all matrix functions from $Ł_{n+1}$ by $\mathcal{L}_{n+1}$. Let P_{n+1} be the set of all $n+1$-valued functions defined on the set V. Then the set of functions R is called *functionally precomplete* (in P_{n+1}) set if an addition to R of a function $f \notin R$ forms the set $\{R, f\}$ *functionally complete*, i.e. if $\{R, f\} = P_{n+1}$. In [1] Bochvar and Finn have proved the set of functions $\mathcal{L}_{n+1}$ is functionally precomplete in P_{n+1} iff n is a prime number. It is shown in [3] that there exists $n+1$-valued logic K_{n+1} that has a non-empty set of tautologies iff n is a prime number, in this case it is proved that $K_{n+1} = \mathcal{L}_{n+1}$. (The proof of the theorem in [3, p. 67] contains a misprint: instead of $x \vee y = (x \overset{k}{\rightarrow} y) \overset{1}{\vee} (y \overset{k}{\rightarrow} x) = max(x, y)$ there should be $x \vee y = (x \overset{k}{\vee} y) \overset{1}{\vee} (y \overset{k}{\vee} x) = max(x, y)$).

Thus in the first case an arbitrary prime number is defined by a precomplete set of functions of related logic and in the second case – by a non-empty set of tautologies of K_{n+1}-logic the functional properties of which agree with those of precomplete logic $Ł_{n+1}$. We offer here a representation of arbitrary precomplete logic $Ł_{n+1}$ by non-empty set of non-precomplete logics $Ł_{n+1}$ and this actually results in one more definition of a prime number, namely, in the form of rooted tree.

2. Partition of a set of Łukasiewicz's logics $Ł_{n+1}$ into equivalence classes

It follows from [1] that if there are values i/n from $V = \{0, 1/n, \ldots, n-1\}$ such that their numerator and denominator are not reciprocal prime numbers and $0 < i < n$ then these values are responsible for non-precompleteness of $Ł_{n+1}$. Therefore they must be remoted and the set V must be reconstructed. This procedure must be repeated until $Ł_{p+1}$ is built where p is a prime number. Hence it is necessary to find the number of m in the row $1, 2, \ldots, n-1$ reciprocal with n and to add 2 since $0 < i < n$. It follows from the definition of Euler's function $\varphi(n)$ in [6] that m is the value of $\varphi(n)$. A convenient formula for calculation of $\varphi(n)$ is contained in [2]: $\varphi(1) = 1, \varphi(pn) = p\varphi(n)$, if $p|n$, and $\varphi(pn) = (p-1)\varphi(n)$ if otherwise.

The construction of precomplete logic $Ł_{p+1}$ from arbitrary logic $Ł_{n+1}$ is reduced then to reprocessing of an arbitrary number n into p, the result of $\varphi(n)$ being added by 1 each time. Let $\varphi^*(n) = \varphi(n) + 1$. It should be noted that in virtue of properties of $\varphi(n)$, $\varphi^*(p) = p$, where p is a prime number, that is p is reprocessed into p. Thus we have an algorithm by which an arbitrary natural number n is reprocessed into a prime number p (consequently $Ł_{n+1}$ into $Ł_{p+1}$):

1. n
2. $\varphi_1^*(n) = n$, if $n = p$ or $\varphi_1^*(n) = m_1$ where $m_1 < n$.
3. $\varphi_2^*(m_1) = m_1$ if $m_1 = p_j$ or $\varphi_2^*(m_1) = m_2$, where $m_2 < m_1$.

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k. $\varphi_{k-1}^*(m_{k-2}) = m_{k-2} = p_1$,

where $i > j > 1$.

Following to this algorithm, for example, Łukasiewicz's logic $Ł_{139}$ is reconstructed into precomplete logic $Ł_{14}$, in this case $k = 4$. It follows from the above algorithm that $\varphi^*_k(n)$ induces a partition of a set of logics $Ł_{n+1}$ into equivalence classes by the relation $\simeq$, where $L_{n_1+1} \simeq\simeq L_{n_2+1}$ iff $\varphi^*_{k_i}(n_1) = \varphi^*_{k_j}(n_2)$ where $1 \leq i,\ j \leq k$. Hence any equivalence class contains one and only one precomplete logic $Ł_{p+1}$. Let $X_{p_1+1}, \ldots, X_{p_s+1}, \ldots$ be equivalence classes where p_s is s-th prime number. For example $X_{p_s+1} = \{6, 9, 11, 13\}$. The question suggests itself: what is the power of every class X_{p_s+1}?

3. Graphs for prime numbers

Since the partition of natural series of numbers forms the basis of partition of a set of logics $Ł_{n+1}$ into equivalence classes then there arises in connection with the question in view a problem of constructing class X_p for an arbitrary prime number p. For this end we must define a function inverse to Euler's function which is designated by $\varphi^{-1}(m)$ and is defined by relation $\varphi^{-1}(m) = \{n : \varphi(n) = m\}$. For example, if $\varphi(n) = 4$ then the above equation has only four solutions, that is $\varphi^{-1}(4) = \{5, 8, 10, 12\}$. The properties of $\varphi^{-1}(m)$ are investigated in [2] where the lower and the upper bounds for any non-empty set of values of $\varphi^{-1}(m)$ are defined. It should be noted that value set of $\varphi^{-1}(m)$ is empty for all odd values of $m > 1$ and also for many even values of m. What is essential is that in [2] the author offers an effective method of constructing the value set od $\varphi^{-1}(m)$ using any m which is the value of $\varphi(n)$. In principle this makes it possible to construct an algorithm which builds according to any prime number its equivalence class X_p. The concept of algorithm consists in the following:

1. p is a prime number.
2. $p - 1$
3. $\varphi_1^{-1}(p-1) = \{\nu_e\}_1 \cup \{\nu_0\}_1$ where $\{\nu_e\}_1$ is a set of even values and $\{\nu_0\}_1$ is a set of odd values. If $\{\nu_0\}_1$ is empty then the equivalence class is built as in the above example when $p = 5$. If otherwise then
4. Every ν_0 is substracted by 1, that is we have $\{\nu_0 - 1\}_1$.
5. $\varphi_2^{-1}(\{\nu_0 - 1\}_1) = \{\nu_e\}_2 \cup \{\nu_0\}_2$.

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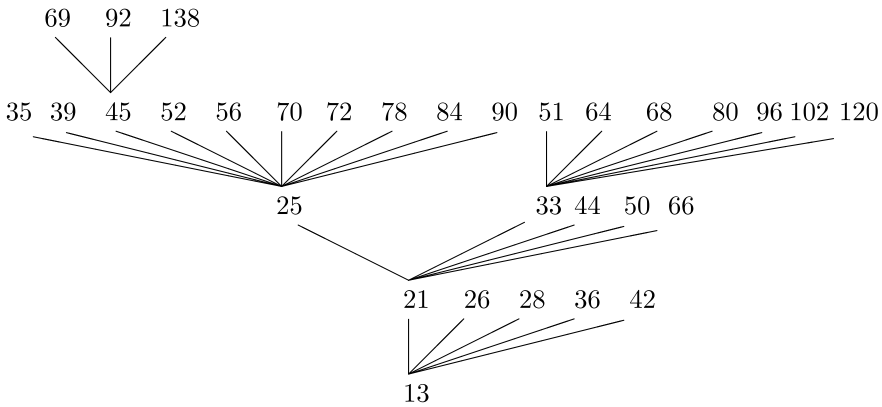
If elements of $\{\nu_0 - 1\}\}$ are not the values of $\varphi(n)$ then the corresponding equivalence class is built since then $\varphi^{-1}(\{\nu_0 - 1\}) = \emptyset$. If the number of even numbers which are not the values of $\varphi(n)$ were finite, then *all* classes X_p starting from some p would have infinite power. It follows from the result of [5] however that there exists an infinite set in even numbers which are not values of $\varphi(n)$. Thus, the necessary condition for finiteness of every class X_p is found. The question on sufficiency is open to discussion.

The algorithm for the construction of equivalence classes X_p using arbitrary p gives us a way of representing prime numbers in the form of rooted tree which we designate by T_p where p is a root and the set of elements X_p is a set of tops. For example, let $p = 13$:



Graphs for the first fifty prime numbers are finite. The hypothesis consists in that every rooted tree T_p is finite. Its proof would have a definite sense for number theory.

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