ON THE DEGREE OF MAXIMALITY OF DEFINITIONALLY COMPLETE LOGICS

The following definition is due to Prof. R. Wójcicki:

A propositional logic \((S, C)\) is definitionally complete iff there is a definitionally complete algebra \(A\) such that for some set of elements \(\{D_i : i \in I\}\) of \(A, C = K^\subseteq\), where \(K = \{(A, D_i) : i \in I\}\).

The consequence operation determined by a class of matrices \(K\) we shall denote, for convenience, by \(K^\subseteq\) (it is usually denoted by \(C_{K^\subseteq}\), see e.g. [6]).

Recall that:

(i) a propositional logic \((S, C)\) is a couple such that \(S\) is a propositional language and \(C\) is a structural consequence on it (see e.g. [6]),
(ii) a finite algebra \(A\) is definitionally complete (or prime, see [1]) iff every function on it is a polynomial,
(iii) the degree of maximality of a logic \((S, C)\) is the cardinal number of all structural strengthenings of \(C\) (R. Wójcicki [7]).

The objective of this note is to prove the following

**THEOREM.** The degree of maximality of a definitionally complete logic is finite.

Our proof will be based on the following Lemma:

**LEMMA 1.** (R. Wójcicki [8]). Let \((S, K^\subseteq)\) be a strongly finite logic. Then the degree of maximality of \((S, K^\subseteq)\) is finite iff the cardinality of the following set

\[
\{K_0^\subseteq : K_0 \subseteq SP_f(K)\}
\]

(where \(P_f\) stands for the operation of forming finite products) is finite.
The fact that every definitionally complete logic is strongly finite is a straightforward corollary to the definition.

In what follows we shall state the submatrices of finite products of $K = \{A_i, D_i\} : i \in I$, $A$ being a definitionally complete algebra, and then prove that they determine a finite number of different consequence operations.

**Lemma 2.** If $A$ is a definitionally complete algebra, then every non-empty subalgebra of a power $A^I$ of $A$ contains the diagonal algebra of $A^I$, $DA^I$, and hence $DA^I$ (which is isomorphic to $A$) is the least subalgebra of $A^I$.

**Proof.** Suppose $B$ is a subalgebra of $A^I$ and there is an element $a = (a_i, i \in I)$ of the diagonal of $A^I$ which is not an element of $B$. Since $A$ is definitionally complete, there is a polynomial $\varphi^A_a$ such that for every element $x$ of $A$, $\varphi^A_a(x) = a$. Consider the polynomial $\varphi^A_a$ on $A^I$. We have that for every element $y$ of $B$, $[\varphi^A_a(y)]_i = a$, and hence $B$ is not a subalgebra of $A^I$.

**Remark.** As it can be seen from the above proof, the assumption of Lemma 2 can be weakened, it is sufficient to assume that every element of $A$ is definable by a polynomial on $A$ (i.e. for every element $a$ of $A$ there is such a polynomial $\varphi^A_a(x)$ on $A$ that $\varphi^A_a(x) = a$ for every $x \in A$).

**Lemma 3.** If $A$ is a definitionally complete algebra, then each non-empty subalgebra of the power $A^n$, $n \in \omega$, is (up to isomorphism) of one of the following forms:

- $DA^n$ (the diagonal algebra of $A^n$, which is isomorphic to $A$),
- $DA^{n-1}xA$,
- $\ldots$
- $DA^2xA^n-2$
- $A^n$.

**Proof.** The fact that $DA^n$ is a subalgebra of $A^n$ follows from Lemma 2. Suppose $B$ is a subalgebra of $A^n$ and let $(b,b,\ldots, b, c) \in B$. Since $A$ is definitionally complete, then for each element $a_i$ of $A$ there is a polynomial $\varphi^A_{a_i}$ such that
\[ \varphi^A_{a_i}(X) = \begin{cases} a_i, & \text{if } x = c \\ x, & \text{otherwise}. \end{cases} \]

Since \( B \) is a subalgebra of \( A^n \), then for each \( a_i \in A \), \( \varphi^A_{a_i}((b, b, \ldots, c)) = (b, b, \ldots, a_i) \in B \). So we obtain that if an element \( (b, b, \ldots, c, c) \), which differs only on the \( n \)-th coordinate from the element of the diagonal \( (b, b, \ldots, b, b) \), belongs to the subalgebra \( B \) of \( A^n \), then each element of \( A^n \) which differs from it on the \( n \)-th coordinate only belongs to \( B \) as well. Clearly, the above argument can be carried out for each coordinate, and hence the Lemma follows.

**Lemma 4.** (M. Maduch [2].) Let \( M_i, i \in I \) be matrices for \( S \) and let \( M = \prod_i M_i \). Then for every set of formulas \( X \)

\[ M^\equiv(X) = \begin{cases} \bigcap_i M_i^\equiv(X), & \text{if } X \text{ is satisfable in all } M; \\ S, & \text{otherwise}. \end{cases} \]

**Lemma 5.** Let \( K \) be a finite set of matrices of the form \( (A, D_i), i = 1, \ldots, n \), where \( A \) is a definitionally complete algebra and \( D_i \subseteq A, i = 1, \ldots, n \). Then the set \( \{ K_0^\equiv : K_0 \subseteq SP_f(K) \} \) is finite.

**Proof.** Without loosing the generality of the proof we can assume, for mere convenience, that the elements of \( \{ D_i : i = 1, \ldots, n \} \) are irreducible. Denote by \( P \) the set of all such products of all subsets of \( K \cup \{(A, D_i \cap \ldots \cap D_k) : 2 \leq k \leq n \} \) that each element does occur only once\(^1\).

We shall prove the Lemma by induction on the number of occurrences of elements from \( K \) in products from \( P_f(K) \).

Let \( M' \in P \) and \( N' \in P \). Consider \( M' \times N' \). If \( N' \) is not one of the elements of the product \( M' \), then \( M' \times N' \in P \).

1. Suppose \( N' \) does occur in the product \( M' \). We shall show that for each submatrix \( N' \) of \( M' \times N' \) there is such a submatrix \( M \) of a matrix from \( P \) that \( N' \equiv M^\equiv \).

Suppose

- \( M_p = (A, D_{j_1}) \times \ldots \times (A, D_{j_k}) \),
- \( N' = (A, D_{j_k}) \).

\(^1\) As a matter of fact we should have considered a set \( P' = P \cup \{ \tau_1 \} \), where \( \tau_1 \) is the unite matrix but since what we really want to know is whether the degree of maximality is finite or not, we can omit \( \tau_1 \), which, incidentally, produces only one new consequence operation; the trivial one.
According to Lemma 3 every non-empty submatrix \( N \) of \( M' \times N' \) is of one of the following forms (up to isomorphism):

(i) \( M \times N' \), where \( M \) is a submatrix of \( M' \),

(ii) \( M_1 \times D(N' \times N') \), where \( M_1 \) is a submatrix of \( M_p \) and \( D(N' \times N') \) is the diagonal matrix of \( N' \times N' \) (which is isomorphic to \( N' \)),

(iii) \( D(M' \times N') \), i.e. the diagonal matrix of the product \( M' \times N' \), defined as follows, \( D(M' \times N') = (DA^{k+2}, DA^{k+2} \cap (\cap_{i=1}^k D_{j_i} \times D_{j_0} \times D_{j_0})) \).

Clearly, \( D(M' \times N') \cong (A, \cap_{i=1}^k D_{j_i} \cap D_{j_0} \cap D_{j_0}) \in \mathcal{P} \).

The case (ii) is trivial.

Now, consider (i). We can distinguish two situations:

(a) there is such a coordinate \( J - i \neq j_0 \), that for each element \( a = (a_{j_1}, \ldots, a_{j_k}, a_{j_0}) \) of \( M \) \( a_{j_i} = a_{j_0} \),

(b) if \( a = (a_{j_1}, \ldots, a_{j_k}, a_{j_0}) \) is an element of \( M \) then every \( b \) which differs from \( a \) on the \( j_i \)-th coordinate only, is an element of \( M \) as well.

Suppose (b) is the case. Then \( M \) is isomorphic to a matrix of the form \( M_1 \times N' \), where \( M_1 \) is a submatrix of \( M_p \). Then, from Lemma 4, we have
\[
(M_1 \times N' \times N')^e = (M_1 \times N')^e.
\]

Now, suppose (a). Then \( M \) is isomorphic to a matrix of the form \( M_2 \times D(N_2 \times N') \), where \( N_2 \) \( \in \mathcal{P} \). Let \( N_2 = (A, D_{j_m}) \). Then \( D(N_2 \times N') \cong (A, D_{j_m} \cap D_{j_0} \cap D_{j_0}) \in \mathcal{P} \) and \( N' \) does not occur in the product of which \( M_2 \) is a submatrix. If \( (A, D_{j_m} \cap D_{j_0}) \) does occur in the product of which \( M_2 \) is a submatrix we can get rid of it in the same manner as of \( \mathcal{N}' \).

Finally, we obtain a matrix \( M_0 \) which is a submatrix of an element of \( \mathcal{P} \).

2. Suppose we have proved the lemma for \( n \) occurrences of \( \mathcal{N}' \). If \( \mathcal{N}' \) occurs in the product \( n + 1 \) times, i.e. if it is of the form \( M_p \times N'^{n+1} \), then it has submatrices of the following form (up to isomorphism):

(i) \( M \times \mathcal{N}' \), where \( M \) is a submatrix of \( M_p \times N'^n \)

or

(ii) \( D(M_p \times N'^{n+1}) \).

Surely, \( D(M_p \times N'^{n+1}) \cong D(M_p \times \mathcal{N}') \) and we can apply the inductive assumption. In the case (i) there are two possible situations
(a) $\mathcal{M} = \mathcal{M}' \times N'$; $\mathcal{M}'$ is a submatrix of $\mathcal{M}_p \times N'^{n-1}$, or
(b) $\mathcal{M} = \mathcal{M}_1 \times D(N'^n)$, $\mathcal{M}_1$ is a submatrix of $\mathcal{M}_p$.

If (a) is the case, then we have from Lemma 4,

$$(\mathcal{M}' \times N'^{n})|_{\mathcal{M}} = (\mathcal{M}' \times N')|_{\mathcal{M}}.$$

In the case (b), since $D(N'^n) \cong N'$, we get

$$\mathcal{M}_1 \times D(N'^n) \times N' \cong \mathcal{M}_1 \times N' \times N',$$

and, as above

$$(\mathcal{M}_1 \times N'^{n})|_{\mathcal{M}} = (\mathcal{M}_1 \times N')|_{\mathcal{M}}.$$

This, the proof is completed.

From Lemma 1 and 5 the proof of the theorem follows.

The following Lemma is to be found in [5]:

**Lemma 6.** (P. Wojtylak). Let $K$ be a class of matrices similar to $S$ and let $C = K|_{C'}$. Then for every structural strengthening $C'$ of $C$, $C' \cong K|_{C'}$ for some $K_0 \subseteq SP(K)$.

We have also the following simple fact:

**Fact.** Let $\mathcal{M}$ be a matrix similar to $S$ and $\mathcal{N}$ be a submatrix of $\mathcal{M}$. Then $\mathcal{M}|_{\mathcal{N}} \leq \mathcal{N}|_{\mathcal{N}}$.

From Lemma 2 and 6 and the above Fact we have the following corollary

**Corollary 1.** Let $A$ be a definitionally complete algebra similar to $S$ and let $C = (A, D)|_{C'}$, $D \subseteq A$. Then the logic $(S, C)$ is maximal.

As a matter of fact, in accordance with the remark to Lemma 2, we have the following stronger corollary

**Corollary 2.** Let $A$ be an algebra similar to $S$ such that every element of it is definable by a polynomial on $A$ and let $C = (A, D)|_{C'}$, $D \subseteq A$. Then the logic $(S, C)$ is maximal.

The above result was proved in a different way by M. Tokarz, [4].

The proof of Lemma 5 suggests the way of determine the degree of maximality of a given definitionally complete logic $(S, C)$. 
We can notice the following observation:

Let \((S, C)\) be a definitionally complete logic and let \(P\) be defined as in the proof of Lemma 5 (now, we need \(\tau_1\) in \(P\)). Then for each logic \((S, C')\) such that \(C' \geq C\) there is a subset \(K_0\) of \(P\) such that \(C' = K_0\). So, if \(K_0 = \{M_1, \ldots, M_k\}\), then \(C' = M_1^\wedge \cap \ldots \cap M_k^\wedge\), where \(\cap\) is the lattice meet.

As an example we shall calculate the degree of maximality of the logic \((S, K \cap dK)\), where \((S, K)\) is classical logic and \((S, dK)\) the logic dual to \((S, K)\) (i.e. it is the logic determined by two-element Boolean algebra with 0 as the designated element). We have

\[ K \cap dK = (B_2, \{0\}, \{1\})^\wedge, \]

where \(B_2\) is the two-element Boolean algebra. So, in \(P\) we have the following matrices

\[ M_1 = (B_2, \{1\}), \]
\[ M_2 = (B_2, \{0\}), \]
\[ M_3 = (B_2, \{1\} \cap \{0\}) = (B_2, \emptyset), \]
\[ M_4 = (B_2, \{1\}) = M_2 \times M_1 \cong M_1 \times M_2 = (B_2, \{10\}), \tau_1, \]

and \(M_4\) has no proper submatrices.

We get the following consequence operations:

1. \(M_1^\wedge = K\).
2. \(M_2^\wedge = dK\).
3. \(\{M_1, M_2\}^\wedge = M_1^\wedge \cap M_2^\wedge = K \cap dK\).
4. \(M_3^\wedge = S_0\), where for every \(X \subseteq S, S_0(X) = \begin{cases} \emptyset, & \text{if } X = \emptyset, \\ S, & \text{otherwise.} \end{cases}\)
5. \(M_4^\wedge = C_1\), where for every \(X \subseteq S, C_1(X) = \begin{cases} \emptyset, & \text{if } X = \emptyset, \\ K \cap dK(X), & \text{if } K(X) \neq S \neq dK(X), \\ S, & \text{otherwise,} \end{cases}\)
   (see Lemma 4).
6. \(\tau_1^\wedge = S\), where \(S(X) = S\) for every \(X \subseteq S\).
7. \(\{M_1, M_3\}^\wedge = K \cap S_0\).
8. \(\{M_1, M_4\}^\wedge = K \cap C_1\).

\[ ^2 \text{Now, we make use of lemma stronger than Lemma 1, i.e. that every structural strengthening of a strongly finite logic is determined by some } K_0 \subseteq SP_f(K) \text{ (for a proof, see [8]).} \]
9. \( \{M_2, M_3\} \models dK \cap S_0 \).
10. \( \{M_2, M_4\} \models dK \cap C_1 \).
11. \( \{M_3, M_4\} \models S_0 \cap C_1 = C_1 \).

An easy inspection shows that if we consider other cases (i.e., consequences determined by three or matrices or subsets of \( P \) containing \( \tau_1 \)) no new consequence operations will be produced. It is quite easily seen that the first 10-th consequence operations are distinct, so the degree of maximality of \( K \cap dK \) is 10. A closer inspection shows that they form the following lattice (see [3], where the degree of maximality of \( K \cap dK \) was originally studied through by different methods):

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\begin{center}
\begin{tikzpicture}
  \node (S) at (0,4) {S};
  \node (K) at (-2,2) {K};
  \node (S0) at (0,2) {S_0};
  \node (dK) at (2,2) {dK};
  \node (KcapS0) at (0,0) {K \cap S_0};
  \node (C1) at (-1,-1) {C_1};
  \node (dKcapS0) at (0,-1) {dK \cap S_0};
  \node (KcapC1) at (-2,-2) {K \cap C_1};
  \node (dKcapC1) at (0,-2) {dK \cap C_1};
  \node (KcapdK) at (-2,-3) {K \cap dK};

  \draw[->] (S) -- (K);
  \draw[->] (S) -- (S0);
  \draw[->] (S) -- (dK);
  \draw[->] (K) -- (S0);
  \draw[->] (K) -- (dK);
  \draw[->] (K) -- (KcapS0);
  \draw[->] (S0) -- (C1);
  \draw[->] (dK) -- (dKcapS0);
  \draw[->] (KcapS0) -- (KcapC1);
  \draw[->] (dKcapS0) -- (dKcapC1);
  \draw[->] (KcapC1) -- (KcapdK);
\end{tikzpicture}
\end{center}
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References


*Section of Logic*

*Institute of Philosophy and Sociology*

*Polish Academy of Sciences*