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ADEQUATE SEMANTICS FOR NON-PSEUDOAXIOMATIC CONSEQUENCE OPERATION

The work refers to W. A. Pogorzelski and P. Wojtylak’s construction of consequence operation defined by quasi algebra (cf. [2], Chapter II). We propose, following J. Czelakowski’s suggestion, a modification of the definition of a $q$-filter, which enables us to generate the class of all non-pseudoaxiomatic consequence operations (Theorem 2).

1. By a quasi ordered algebra for a language $S = < S, F_1, \ldots, F_n >$ freely generated by a set of propositional variables $V = \{ p_1, p_2, \ldots \}$ (shortly, $q$-algebra), we understand any pair of the form $A = < A, R >$, where $A = < A, f_1, \ldots, f_n >$ is an algebra similar to $S$, and $R$ is a quasi-ordering on $A$. A consequence $C$ is said to be pseudoaxiomatic (cf. 1) iff it is structural and $C(\emptyset) \neq \emptyset$. Theorem 1 ([2], p.75). If $C$ is a structural finitistic consequence operation and $C(\emptyset) \neq \emptyset$, then there exists a $q$-algebra $A$, such that $C = C^q_A$.

2. Let $A$ be any $q$-algebra for the language $S$. Definition 1 comes from [2]:

**Definition 1.** A non-empty set $B \subseteq A$ is a $q$-filter in $A$ iff

\[
\forall H \in Fin(B), B_n(B_1(H)) \subseteq B, \text{ where for every } X \subseteq A
\]

\[
B_n(X) = \{ a \in A : bRa \text{ for each } b \in X \}
\]

\[
B_1(X) = \{ a \in A : aRb \text{ for each } b \in X \}.
\]

The set of all $q$-filters in $A$ and a consequence operation generated by a generalized matrix $M = < A, q\mathcal{F}(A) >$ are denoted by $q\mathcal{F}(A)$ and $C^q_A$ respectively.
3. We modify the notion of a $q$-filter.

**Definition 2.** A set $B \subseteq A$ is a $q^*$-filter iff $B_u(B(B)) \subseteq B$.

Let $q^*\mathcal{F}(A)$ denote the set of all $q^*$-filters and $C^q_A$ a consequence operation generated by $M = \langle A, q^*\mathcal{F}(A) \rangle$.

**Fact 1.** $C^q_A$ is a structural consequence.

**Fact 2.** $C^q_A(\emptyset) = \bigcap\{\alpha \in S : h(\alpha) \in B_u(A), \text{ for every } h \in \text{Hom}(S, A)\}$.

**Lemma 1.** Let $M = \langle A, D \rangle$ be any matrix with $D \neq \emptyset$; then there exists a quasi ordering on $A$, such that:

$$C_M = C^q_A.$$

We shall say that a $q$-algebra $A$ is strongly adequate for the consequence $C$ provided that $C = C^q_A$. Let us denote the class of all consequences having strongly adequate $q$-algebra by $\zeta^q$.

**Fact 3.** $C_0 \notin \zeta^q$, where $C_0$ is the almost-inconsistent consequence.

For any family $A_i = \langle A_i, R_i \rangle$ ($i \in I$) of $q$-algebras, we define the product of $q$-algebras $\bigcap A_i$ is a $q$-algebra.

**Lemma 2.** If $\{A_i\}_{i \in I}$ is a family of $q$-algebras, then

$$C^q_{\bigcap A_i} = \inf C^q_{A_i}.$$

Using the properties mentioned above we shall prove a theorem which characterizes the class $\zeta^q$.

**Theorem 2.** $C \in \zeta^q$ iff $C$ is not a pseudoaxiomatic consequence.

**Proof.** ($\Rightarrow$) Let $C \in \zeta^q$, thus on the basis of Fact 1 it is enough to show that $C(\emptyset) = \bigcap\{C(X) : \emptyset \neq X \subseteq S\}$.

The inclusion from left to right is obvious. To prove the converse let us suppose that there exists a formula $\beta \in S$ such that:

1. $\beta \in \bigcap\{C(X) : \emptyset \neq X \subseteq S\}$

and
(2) \( \beta \not\in C(\emptyset) \).

But \( C \in \zeta^q \) and therefore

(3) \( C = C^q_A \) for some \( q \)-algebra \( A = < A, R > \).

Thus from Fact 2 there exist \( h \in Hom(S, A) \) and \( a_0 \in A \) such that

(4) \( non(a_0Rh(\beta)) \).

Let \( B_0 = \{ x : a_0Rx \} \) and \( p_0 \in V - V(\beta) \) (\( V(\beta) \) denotes the set of all propositional variables occurring in the formula \( \beta \)). We define a mapping \( h_1 : V \to A \) in the following way:

\[
h_1(p_i) = \begin{cases} a_0 & \text{iff } p_i = p_0 \\ h(p_i) & \text{iff } p_i \neq p_0 \end{cases}
\]

and we extend it to \( H \in Hom(S, A) \). In this situation

(5) \( B_0 \in q^*F(A) \),

(6)

(7)

Thus \( \beta \not\in C^q_A(\{p_0\}) \) which contradicts (1).

(\( \Leftarrow \)) Assume that \( C \) is a structural consequence and let \( C(\emptyset) = \cap \{ C(X) : \emptyset \neq X \subseteq S \} \). Put \( \mathcal{M} = < S, C(X) : \emptyset \neq X \subseteq S > \). It determines the family of matrices of the form

(1) \( M_X = < S, C(X) > \) for every \( \emptyset \neq X \subseteq S \)

in the evident way.

But, according to Lemma 1, for every \( M_X \) there is a \( q \)-algebra \( A_X = < S, R_X > \) such that

(2) \( C^q_X = C_{M_X} \).

Let us put

(3) \( A = \cap A_X \).

We shall show that \( A \) is a strongly adequate \( q \)-algebra for \( C \), i.e. that

(4) \( C^q_A(Y) = C(Y) \) for all \( Y \subseteq S \).

(\( \subseteq \)) Let us consider two situations.

1) \( Y \neq \emptyset \), so

\[
C^q_A(Y)^{Lemma 2} \cap_{X \neq \emptyset} C^q_{A_X}(Y) \subseteq C^q_{A_Y}(Y)^{Lemma 1}C_{M_Y}(Y) = C(Y)
\]

2) \( Y = \emptyset \), then
Let us suppose that there exist \( \alpha \in S \) and \( Y \subseteq S \) such that
1) \( \alpha \in C(Y) \)

and
2) \( \alpha \notin C_{A_X}^q(Y) \).

It results from Lemma 2 that
3) \( \alpha \notin C_{A_X}^q(Y) \) for some \( X \neq \emptyset \).

Therefore, according to Lemma 1, because \( C_{A_X}^q = C_{M_X} \), we get that there exists \( h \in \text{End}(S,S) \) such that
4) \( h(Y) \subseteq C(X) \)

and
5) \( h(\alpha) \notin C(X) \).

Since \( C \) is structural from 1) we get:
6) \( h(\alpha) \in h(C(Y)) \subseteq C(h(Y)) \subseteq C(C(X)) = C(X) \)

which contradicts 5).

References


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