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ON SOME METHOD OF AXIOMATIZATION OF SOME PROPOSITIONAL CALCULI

In this paper* a simple method for the finite axiomatization of some propositional logics determined by finite matrices is given. Particular examples of such logics are the finite valued: Łukasiewicz's logics, modal logics containing $S4$, and intermediate logics in various connectives versions.

Let Fm be the set of formulas formed in the usual manner by means of the infinite set $\{p_0, p_1, \ldots\}$ of propositional variables and the finite set Con of connectives. For $C \in Con$ let $ar(C)$ be the arity of C. Formulas are denoted by $a, b, c, \ldots$. By Fm^k we mean the restriction of Fm into the variables $p_o, \ldots, p_{k-1}$. We assume that in terms of the connectives of Con, two peculiar (but not necessarily different) binary connectives "$\supset$" and "$\rightarrow$", which we call implications, are defined. By Sb, MP, Cn we denote the consequence operators based on substitution, modus ponens for "$\supset$" as well as substitution and modus ponens for " " respectively. Then for every $X \subseteq Fm, Cn(X) = MP(Sb(X))$ (cf. eg. [1]). The letters M, N denote logical matrices of the type of Con, and by $|M|$ we mean the number of truth-values of M. The set of all tautologies of M is denote by $E(M)$ and $E^k(M) = E(M) \cap Fm^k$. It is well-known that the set $E(M)$ is closed under Sb.

Lemma 1. *For every finite matrix N with $|N| \leq k$: If $E(M) = E(N) \neq \emptyset$ then $E^k(M) - E(N) \neq \emptyset$.*

By ax_0 we denote the set of the following formulas:

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$F1$ $p_0 \supset (p_1 \supset p_0)$
$F2$ $(p_0 \supset (p_1 \supset p_2)) \supset ((p_0 \supset p_1) \supset (p_0 \supset p_2))$
$F3$ $p_0 \to p_0$
$F4$ $(p_0 \to p_1) \supset ((p_1 \to p_2) \supset (p_0 \to p_2))$
$F5$ $(p_0 \to p_1) \supset (p_0 \supset p_1)$
F^k_C $(p_0 \to p_m) \supset ((p_m \to p_0) \supset (C(p_1, \dots, p_{ar(C)}) \to C(p_1, \dots, p_{m-1}, p_0, p_{m+1}, \dots, p_{ar(C)})$

where $C \in Con$ and $1 \leq m \leq ar(C)$.

We are interested in the axiomatization of matrices which fulfill the following condition:

$(*)$ $ax_0 \subseteq E(M)$ and $E(M)$ is closed under modus ponens for "$\supset$". The formulas $F1$, $F2$ form an axiomatic of the positive implicational Hilbert calculus. Thus, if matrix M fulfills $(*)$ then in its tautologies having the form $a_1 \supset (a_2 \supset \dots \supset (a_k \supset a)\dots)$ the positions of the antecedents $a_1, \dots, a_k$ are not essential and multiple antecedents can be omitted. Therefore, we can write such a formula as $\{a_1, \dots, a_k\} \supset a$.

Lemma 2. *If M fulfills $(*)$ and $|M| = n$ then*
F'_n $\{(p_i \to p_j) \supset ((p_j \to p_i) \supset p_0) : 1 \leq i \neq j \leq n+1\} \supset p_0$
is a tautology of M.

Let "$\sim_M$" denote the congruence relation of Fm defined as follows
$a \sim_M b$ iff $va = vb$ for every valuation v in M.

Let $Rep^k(M)$ be a system of representatives of $Fm^k/\sim_M$ containing variables $p_0, \dots, p_{k-1}$. Then for each $C \in Con$, $b_1, \dots, b_{ar(C)} \in Rep^k(M)$ there exists $b \in Rep^k(M)$ such that $C(b_1, \dots, b_{ar(C)}) \to b$, $b \to C(b_1, \dots, b_{ar(C)}) \in E^k(M)$. Let $CF^k(M)$ denote the set of all those implications. (Since for every k $Fm^k/\sim_M$ is finite then obviously $CF^k(M)$ is also finite).

Lemma 3. *If M fulfills $(*)$ then $E^k(M) \subseteq Cn(ax_0 \cup CF^k(M))$.*

The proof can be found by examining the proofs of Theorems 13 and 14 in [8].

Theorem. *If M fulfills $(*)$ and $|M| = n$, then $E(M) = Cn(ax(M))$, where $ax(M) = ax_0 \cup CF^n(M) \cup \{F'_n\}$.*

Proof. Suppose indirectly that
(1) $a \in E(M) - Cn(ax(M))$

By Lindebaum's well-known property (cf. [2], [1]) there exists such a set of formulas $X = MP(Sb(ax(M)) \cup X)$ that

(2) $a \notin X$ and $a \in MP(X \cup \{b\})$ for every $b \notin X$

Let "$\sim_X$" be the relation defined as folloows

$b \sim_X c$ iff $b \to c,\ c \to b \in X$

By $F3, F4$ and F_C^k this relation is a congruence of Fm. Then $N = ((Fm/\sim_X, Con/\sim_X), Cn(ax(M))/\sim_X)$ is a logical matrix such that

(3) $Cn(ax(M)) \subseteq E(N)$

$a \notin E(N)$

Now we shall prove that

(4) $|N| \leq n$

Suppose indirectly that there exists such a valuation v in N that if $1 \leq i \neq j \leq n+1$ then $vp_i \neq vp_j$. Let $a_1, \ldots, a_{n+1}$ be formulas such that $vp = |a_1| \sim_X, \ldots, vp_{n+1} = |a_{n+1}| \sim_X$. Then $a_i \to a_j \notin X$ or $a_j \to a_i \notin X$. Hence by (2) we have $a \in MP(X \cup \{a_i \to a_j, a_j \to a_i\})$. Since the deduction theorem holds for the Hilbert's calculus $(a_i \to a_j) \supset ((a_j \to a_i) \supset a) \in X$. Hence, using F_n' we obtain $a \in X$. A contradiction. Therefore (4) holds.

Let us go back to the main part of this proof. By (1) and (3) we have $E(M) - E(N) \neq \emptyset$. Hence by (4) and Lemma 1 we obtain $E^n(M) - E(N) \neq \emptyset$. Then by Lemma 3 we have $Cn(ax(M)) \nsubseteq E(N)$. A contradiction. So $E(M) \subseteq Cn(ax(M))$.

By Lemma 2 and the definition of $ax(M)$ we obtain $Cn(ax(M)) \subseteq E(M)$.

By observation of the condition $(*)$ it follows that finding a finite axiomatization for many finite logics can be a simple consequence of the well-known deduction theorems and theorems about congruences for these logics. For example these theorems hold for intermediate logics, modal logics containing $S4$ (for which the deduction theorem has already been proved in [5]) and Łukasiewicz's logics (for which the deduction theorem has already been proved in [6] and [4]).

Examples. I. Let Int_Δ be the intuitionistic propositional logic in the set Δ of connectives such that

$$\Rightarrow \in \Delta \subseteq \{\Rightarrow, \wedge, \vee, \equiv, \neg\}$$

and let M_Δ be a finite matrix of type of Δ for an intermediate logic, i.e. $E(M_\Delta) \supseteq Int_\Delta$. Then M_Δ with $\supset = \rightarrow = \Rightarrow$ fulfills $(*)$ and therefore $ax(M_\Delta)$ together with substitution and modus ponens for "$\Rightarrow$" forms a finite axiomatization for $E(M_\Delta)$.

II. Let $S4_\Delta$ be the modal propositional logic in the set of connectives such that

$$\Rightarrow, \Box \in \Delta \subseteq \{\Rightarrow, \wedge, \vee, \equiv, \neg, \Box\}$$

and let M_Δ be a finite matrix of type of Δ for a modal logic containing $S4_\Delta$, i.e. $E(M_\Delta) \supseteq S4_\Delta$ and $E(M_\Delta)$ is closed under the rules of modus ponens for "$\Rightarrow$" and necessitation. Then M_Δ with $\supset$ and $\rightarrow$ defined as follows

$$a \supset b = La \Rightarrow b \text{ and } a \rightarrow b = L(a \Rightarrow b)$$

fulfills $(*)$ and therefore $ax(M_\Delta)$ with substitution, modus ponens for "$\Rightarrow$" and necessitation is a finite axiomatization for $E(M_\Delta)$.

III. Let M be any n-valued matrix such that n-valued Łukasiewicz's implication "$\Rightarrow$" is definable in this matrix. Then M with $\rightarrow = \Rightarrow$ and $\supset$ defined as follows

$$a \supset b =$$

fulfills $(*)^\dagger$ and therefore $ex(M)$ with substitution and modus ponens for "$\Rightarrow$" is a finite axiomatization for $E(M)$.

Final remarks. The proof of Theorem was found by examining the methods of axiomatizations of logics which were done by M. Wajsberg in [8], J. Łoś in [2] and G. Asser in [1]. In Wajsberg's work (cf. Note 8) and in [6] similar criteria were given, but I do not know how to use them in order to prove the theorems appearing in Examples. Many partial axiomatizations given in Examples were found earlier. The reviews of axiomatizations for intermediate and Łukasiewicz's logics are contained in [9] and [3]. Worthwhile, it would be to note that axiomatizations of Łukasiewicz's logics in all connectives versions was done by K. Schröter in [7] (where also a theorem saying that for every finite logic can be found a finite axiomatization in Gentzen's style is to be found).

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† In order to find axiomatizations also in versions which do not contain Łukasiewicz's n-valued negation as definable we must use the deduction theorem from [4] where the appropriate implication is defined as here.

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