A NOTE ON IMPLICATIONAL INTERMEDIATE CONSEQUENCES

1. A structural consequence operation $C$ is s.s.c. (strongly structural complete) if $C(\emptyset) = C(\emptyset)$ implies $C' = C$ for each (structural) strengthening $C' \geq C$. $C$ is s.f.a. (strongly finite approximable) if $G$ is determined by a set $K$ of finite matrices; if $K$ itself can chosen to be finite, $C$ is said to be tabular (alias strongly finite).

In the sequel the letters $C, D$ range exclusively over $\to$-intermediate consequences, i.e. consistent structural $C \geq C_H$ where $C_H$ is the intuitionistic consequence restricted to the propositional language $F(\to)$. $C_H$ is simply the smallest structural consequence $C$ in $F(\to)$ such that Modus Ponens holds in $C$ and

$$(dt) \ Q \in C(X, P) \Rightarrow P \to Q \in C(X) \quad (\text{deduction theorem})$$

where $X, Y$ range over formula sets, $P, Q$ over formulas. In Section 2 we will prove the following

**Theorem.** $C$ is s.s.c. iff $C$ is s.f.a. Moreover, each s.s.c. $C$ satisfies $(dt)$.

Let $\triangle$ denote the set of all $C$ satisfying $(dt)$ and $\triangle^\omega = \{ C \in \triangle | C \ \text{finitary} \}$. The $C \in \triangle^\omega$ are just the axiomatic strengthenings of $C_H$. Clearly, $C \in \triangle$ implies $C^\omega \in \triangle^\omega$, where $C$ is the finitary cernel of $C\ [P \in C^\omega(X) \iff (\exists \ \text{finite } Y \subseteq X) \ (P \in C(Y))]$, but the converse may be wrong.

To each $C$ there is a smallest $D$ with $D(\emptyset) = C(\emptyset)$, denoted by $C^\delta$. Obviously $C^\delta \in \triangle^\omega$. There is also the largest $D$ with $D(\emptyset) = C(\emptyset)$ (in which hold all sequential rules admissible for $C(\emptyset)$), denoted by $C^\delta$. Clearly, $C^\delta$ is s.s.c. (the strong structural completion of $C$); moreover, if $D$ is s.s.c.
and $D(\emptyset) = C(\emptyset)$ then $D = C^5$ (see e.g. [2] for these simple facts). Thus, the conditions $C$ is s.s.c, $C = C^5$, each admissible rule for $C(\emptyset)$ holds in $C$, are all equivalent. Notice that the finitary cernel of $C^5$ is structural complete in the finitary sense.

Here some corollaries from the Theorem. The first one is the main result in [4]:

**Corollary 1.** Let $C \in \Delta^\omega$. Then $C$ is s.s.c. iff $C$ is tabular.

**Proof.** If $C$ is tabular then $C$ is trivially s.f.a. hence s.s.c. by the Theorem. If $C$ is non-tabular, consider the s.f.a. $C^\prime = \inf \{D | C(\emptyset) \subseteq D(\emptyset) & D \text{ tabular}\}$. By the Theorem, $C^\prime$ is s.s.c and $C^\prime \in \Delta$. Moreover, $C^\prime(\emptyset) = C(\emptyset)$ since each $\rightarrow$-intermediate logic is tabular approximable ([1]). Thus, $C^\prime = C^5$. As is well known, a non-tabular s.f.a. intermediate $D \in \Delta$ cannot be finitary.* Thus, $C \neq C^5$ because $C$ is finitary while $C^5$ is not.

**Corollary 2.** Each finitary $C$ satisfies ($dt$).

**Proof.** Let $X = \{P_0, \ldots, P_{n-1}\}$. It suffices to show

(1) $P_n \in C(X) \Rightarrow P_0 \rightarrow \ldots \rightarrow P_n \in C(\emptyset)$.

Let $P_n \in C(X)$. Then $P_n \in C^5(X)$. Thus, $P_0 \rightarrow \ldots \rightarrow P_n \in C^5(\emptyset) = C(\emptyset)$ because $C^5 \in \Delta$ by the Theorem.

Hence, the finitary cernel of each $C$ satisfies ($dt$). Of course, this is no argument in favour of a positive answer to

**Question 1.** Does every $C \geq C_H$ satisfy ($dt$)?

It is obvious that the infinitary sequential rule

$$\rho = \{(p_i \rightarrow p_j) \rightarrow (p_j \rightarrow p_i) \rightarrow p_0 | 1 < i < j\}/p_0$$

holds for each s.f.a. and hence s.s.c. $C$. Let $C^\rho$ denote the smallest $D \geq C$ such that $\rho$ holds in $D$. Then $C^\rho \leq C^5$ and all subdirect irreducible models of $C^\rho$ are finite. Moreover, $C^\rho \in \Delta$.

*This nice exercise was communicated to me by A. Wroński in 1978: Put $X = \{(p_i \rightarrow p_j) \rightarrow (p_j \rightarrow p_i) \rightarrow p_0 | 0 < i < j < \omega\}$. Then $p_0 \in D(X)$ but $p_0 \notin D(Y)$ for finite $Y \subseteq X$ (use $dt$ and the fact that $D(\emptyset)$ is determined by subdirect irreducibles). By the theorem, the assumption $D \in \Delta$ is superfluous. We mention that in the presence of $\lor$ the assumption $C \in \Delta$ cannot completely be omitted but replaced by the weaker assumption $C$ is monotonic (because finitary monotonic $C$ are determined by subdirect irreducible algebraic matrices, see [5]).
QUESTION 2. Is $C^δ = C^ρ$ (in particular for $C = C_H$)?

If the answer is no, perhaps some other axiomatization of $C^δ$ with finitely many rules will be found.

**Remark.** The Theorem and its Corollaries fail for $C \geq C_I$ = intuitionistic consequence in the full propositional language. It seems that $C^δ$ is s.f.a. only in particular cases e.g. if $C(\emptyset)$ is locally finite. There are many open questions concerning the $C$ with $C(\emptyset) = C_I(\emptyset)$. E.g., is $C^δ_I$ or its finitary kernel finitely based? Notice that $\Delta$ is a complete sublattice of the lattice of all $C \geq C_I$. Let $C^A$ be the largest $C$ with $C(\emptyset) = C_I(\emptyset)$ which satisfies $(dt)$. Perhaps there is a chance to characterize $C^A$.

2. We will now prove the theorem. Write $P \equiv Q$ for $C_H(P) = C_H(Q)$.

**Lemma 1.** Define $e \in Sb$ by $e : p \mapsto (Q \rightarrow p), p \in Var$. Then $eP \equiv Q \rightarrow P$.

**Proof.** Induction on $P$. Observe $(dt)$.

**Lemma 2.** If $C$ is tabular then $C = C^δ$.

**Proof.** As was shown in [3], $C^δ$ is structural complete in the finitary sense. Since $C$ is finitary, $C \leq C^δ$, hence $C = C^δ$.

**Lemma 3.** Each s.f.a. $C$ is s.s.c.

**Proof.** Let $C = \inf \{C_i | i \in I\}$, each $C_i$ tabular. Suppose $Q \not\in C(X)$ so that $Q \not\in C_k(X)$ for some $k \in I$. Since $C_k$ is tabular, there is, obviously, some $f \in Sb$ such that $Q' \not\in C_k(X')$ with $X' = FX, Q' = fQ$, and $X'$ contains a finite number of variables only. By Diego [1], $C_H(X') = C_H(Y)$ for some finite $Y = \{P_1, \ldots, P_n\}$. Define $e_j \in Sb$ by $e_j : p \mapsto (P_j \rightarrow p)$ and put $e = e_1 \circ \ldots \circ e_n$. Then $eP \equiv P_1 \rightarrow \ldots \rightarrow P_n \rightarrow P$ (Lemma 1), and therefore $eY \subseteq C_H(\emptyset) \subseteq C(\emptyset)$. But $eQ' \equiv P_1 \rightarrow \ldots \rightarrow P_n \rightarrow Q' \not\in C_k(\emptyset)$ since $Q' \not\in C_k(Y)$. Thus, $eQ' \not\in C(\emptyset)$ and for $d = e \circ f$ we obtain $dX \subseteq C(\emptyset)$, but $dQ \not\in C(\emptyset)$. Therefore, $C$ is s.s.c.

**Proof of the Theorem.** Let $C$ be s.s.c. and consider the s.f.a. $C' = \inf \{D | C(D) \subseteq C(\emptyset) & D \text{ tabular} \}$. Then $C'(\emptyset) = C(\emptyset)$ ([1]) and $C'$ is s.s.c. by Lemma 3. Therefore $C = C'$, since there is only one s.s.c. $D$ with a
given set of tautologies. Thus, \( C \) is s.f.a. The converse is just Lemma 3. Now let \( C \) be s.s.c. Then \( C \) is s.f.a. by the proceeding. Hence \( C \in \Delta \), because each tabular \( D \) satisfies \((dt)\) by Lemma 2.

References