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A NEW VARIANT OF THE GÖDEL-MALCEV THEOREM FOR THE CLASSICAL PROPOSITIONAL CALCULUS

The Gödel-Malcev theorem says that every consistent system has a model. The theorem can be formulated as follows:

(1) If X is a consistent system then there exists such a $0-1$ valuation h that $hX \subseteq \{1\}$.

It has the following syntactical versions:

(2) If X is a consistent system then there exists such a substitution s that $sX \subseteq CPC$.

(CPC – the set of all classical theses).
Our variant is a strengthening of (2) and has the following form:

(3) If X is a consistent system then there exists such a substitution s that $X = s^{-1}CPC$.

This theorem can be treated as a *theorem on representation of consistent systems by substitutions.*

Let $Ln = (Fr, \neg, \wedge, \vee, \rightarrow, \equiv)$ be an algebra of formulas formed by means of propositional variables $p_0, p_1, \ldots$ and the connectives: $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication) and $\equiv$ (equivalence). By the symbol $Fr^{(k)}$ $(k = 1, 2, \ldots)$ we denote the set of all formulas by means of variables $p_0, \ldots, p_{k-1}$ (and connectives) and by the symbol $Ln^{(k)}$ the subalgebra $(Fr^{(k)}, \neg, \wedge, \vee, \rightarrow, \equiv)$ of algebra Ln determined by variables $p_0, \ldots, p_{k-1}$. By the symbol Cn we denote an operation of consequence (consequence for short) determined by the classical theses, and the rule of modus ponens defined on all subsets of the set Fr (i.e. $Cn(X) = \bigcap\{Y \subseteq Fr : X \cup CPC \subseteq Y$ and the se Y is closed under the rule of modus ponens} ($X \subseteq Fr$)). A set of formulas X is a system if $X = Cn(X)$. A system X is consistent if $X \neq Fr$. The symbol M denotes the classical $0-1$ matrix. The expressions $Hom(Ln, Ln), Hom(Ln, M), Hom(Ln^{(k)}, Ln^{(k)}), Hom(Ln^{(k)}, M)$ denote sets of homomorphisms between the indicated algebras (in the second and fourth cases it would be more accurate to speak on the algebra of the matrix M). The first and the second set of homomorphisms will be called substitutions and $(0-1)$ valuations, respec. Without further mention we will use the following well-known equivalence: $a \in Cn(X)$ iff for every $h \in Hom(Ln, M)$ if $hX \subseteq \{1\}$ then $ha = 1(a \in Fr, X \subseteq Fr)$.

THEOREM. *If X is a consistent system there exists such a substitution s that $X = s^{-1}CPC$.*

PROOF. If A, B are any subsets of any space and f is any function defined in this space then the following equality holds:

$$A = f^{-1}B \text{ iff } fA \subseteq B \text{ and } f(-A) \subseteq -B.$$

Hence the following conjunction is an equivalent form of the equality from Theorem:

$(*)$ $sX \subseteq CPC$ and $s(Fr - X) \cap CPC = \emptyset$.

Let k be any fixed natural number ≥ 1. Let $x_0, \ldots, x_{2^k-1}, y_0, \ldots, y_{k-1} \in \{0, 1\}$ and let

$$F^k_{(x_0,\ldots,x_{2^k-1})} : \{0,1\}^k \to \{0,1\}$$

be such a function that the following equality is satisfied

1. $F^k_{(x_0,\ldots,x_{2^k-1})}(y_0, \ldots, y_{k-1}) = x_{|y_0,\ldots,y_{k-1}|}$

where $|y_0, \ldots, y_{k-1}| = y_0 2^0 + \ldots + y_{k-1}2^{k-1}$.

Since the classical matrix is functionally complete, the above function is definable in it. To the functions of this type correspond k-ary connectives denoted by the same symbols which are definable by the primitive connectives of the language Ln.

For all $i, j (0 \leq i < k, 0 \leq j < 2^k)$ we pick $x^i_j \in \{0, 1\}$ in such a way that the following equality is satisfied:

2. $|x_j^0, \ldots, x_j^{k-1}| = j$

and let $f_0, \ldots, f_{2^k-1} \in Hom(Ln^{(k)}, M)$ be picked in such a way that the following equality is satisfied:

3. $f_j p_i = x_j^i$.

Note that the homomorphisms $f_0, \ldots, f_{2^k-1}$ are all different $0-1$ valuations for the variables $p_0, \ldots, p_{k-1}$. Since X is a consistent system, then by virtue of the Gödel-Malcev theorem it follows that there exists such a $0-1$ valuation g that $gX \subseteq \{1\}$. Let G be the set of all valuations of this kind and let G_k be the set of all homomorphisms of the type $Ln^{(k)} \to M$ having an extension in G. We pick a sequence $g_0, \ldots, g_{2^k-1}$ G_k in such a way that

4. $G_k = \{g_0, \ldots, g_{2^k-1}\}$.

Let $t \in Hom(Ln^{(k)}, Ln^{(k)})$ satisfy the following condition

5. $tp_i = F^k_{(g_0 p_i, \ldots, g_{2^k-1} p_i)}(p_0, \ldots, p_{k-1})$.

Now we show that

$$g_j p_i = f_j t p_i.$$

The proof of this equality has the following form

$$f_j t p_i =_5 F^k_{(g_0 p_i, \ldots, g_{2^k-1} p_i)}(f_j p_0, \ldots, f_j p_{k-1}) =_3$$

$$=_3 F^k_{(g_0 p_i, \ldots, g_{2^k-1} p_i)}(x_j^0, \ldots, x_j^{k-1}) =_{1,2} g_j p_i.$$

Since f_j, g_j, t are homomorphisms, for every $a \in Fr^{(k)}$ the following equality holds

6. $g_j a = f_j t a$.

Now we prove the following equivalence

7. $ta \in CPC$ iff $g_0 a = \ldots = g_{2^k-1} a = 1$.

($\rightarrow$). Let $g_j a \neq 1$ for some $0 \leq j < 2^k$. By 6 we obtain $f_j t a \neq 1$. Hence $ta \notin CPC$.

(←). Let $g_0 a = \ldots = g_{2^k-1}a = 1$. By 6 we have $f_0 ta = \ldots = f_{2^k-1}ta = 1$. As has been mentioned, $f_0, \ldots, f_{2^k-1}$ are the $0-1$ valuations for the variables $p_0, \ldots, p_{k-1}$. Since $ta \in Fr^{(k)}$ then $ta \in CPC$. In this way the proof of 7 has been completed.

From the definition of G and G_k it follows that

$$g_0(X \cap Fr^{(k)}) = \ldots = g_{2^k-1}(X \cap Fr^{(k)}) \subseteq \{1\}.$$

Hence and 7 we have

$$t(X \cap Fr^{(k)}) \subseteq CPC.$$

Assume, now, that $a \in (Fr-X) \cap Fr^{(k)}$. Since $a \notin Cn(X)$ then there exists such a valuation g that $gX \subseteq \{1\}$ and $ga \neq 1$. Hence by the definition of G we have $g \in G$. Let g' be the restriction of g to the variables $p_0, \ldots, p_{k-1}$. Obviously $g' \in G_k$. Since $g'a \neq 1$ then by virtue of $4, 7$ we have $ta \notin CPC$. So $t((Fr-X) \cap Fr^{(k)}) \cap CPC = \emptyset$. Then

8. $t(X \cap Fr^{(k)}) \subseteq CPC$ and $t((Fr-X) \cap Fr^{(k)}) \cap CPC = \emptyset$.

Next we prove that the substitution t can be extended into a substitution t^+ defined on p_k in such a way that 8 is satisfied for $k+1$. The sets G_k and G_{k+1} are picked in such a way that every homomorphisms belonging to G_k has an extension onto the variable p_k which belongs to G_{k+1} and G_{k+1} includes only such homomorphisms which are extensions of some homomorphisms from G_k. If now $h \in G_k$ then by the symbols h', h" we denote all extensions of h onto the variable p_k that belong to G_{k+1}. (If h has only one extension then $h' = h$"). Hence and by virtue of 4 we obtain

4′. $\{g'_0, \ldots, g"_{2^k-1}, g"_0, \ldots, g"_{2^k-1}\} = G_{k+1}$.

Let, now, $u \in Hom(Ln^{(k+1)}, Ln^{(k+1)})$ satisfy the following condition

5′. $up_i = F^{k+1}_{(g_0 p_i, \ldots, g'_{2^k-1} p_i, g"_0 p_i, \ldots, g"_{2^k-1} p_i)}(p_0, \ldots, p_k) \qquad (0 \leq i \leq k)$

As by 4 and 5 we have been proved 8 (for t and k), in quite analogous way be 4′ and 5′ we prove that

8′. $u(X \cap Fr^{(k+1)} \subseteq CPC$ and $u((Fr-X) \cap Fr^{(k+1)}) \cap CPC = \emptyset$.

Let h be any $0-1$ valuation. By 5 and 5′ we have

$$htp_i = F^k_{(g_0 p_i, \ldots, g_{2^k-1} p_i)}(hp_0, \ldots, hp_{k-1})$$

$$hup_i = F^{k+1}_{(g_0 p_i,\ldots,g'_{2^k-1}p_i,g"_0 p_i,\ldots,g"_{2^k-1}p_i)}(hp_0,\ldots,hp_k) \qquad (0 \le i < k)$$

Note that if the number $|hp_0,\ldots,hp_{k-1}|$ is denoted by the letter j then

$$htp_i = g_j p_i$$
$$hup_i = \begin{cases} g'_j p_i & \text{if} \quad hp_k = 0 \\ g"_j p_i & \text{if} \quad hp_k = 1. \end{cases} \qquad (0 \le i < k)$$

For $i < k$ we have $g_j p_i = g'_j p_i = g"_j p_i$. Then for $i < k$ we have $htp_i = hup_i$. Since h is any $0-1$ valuation and h, t, u are homomorphisms, then

9. $\ulcorner ta = ua \urcorner \in CPC$ for every $a \in Fr^{(k)}$.

Let, now, $t^+ \in Hom(Ln^{(k+1)}, Ln^{(k+1)})$ satisfy the following condition

$$t^+ p_i = \begin{cases} tp_i & \text{if} \quad i < k \\ up_k & \text{if} \quad i = k. \end{cases} \qquad (0 \le i \le k)$$

Hence and by 8′, 9 it is easy to see that t^+ is an extension of t onto the variable p_k such that

8". $t^+(X \cap F^{(k+1)}) \subseteq CPC$ and $t^+((Fr - X) \cap Fr^{(k+1)}) \cap CPC = \emptyset$

Now we come to the final part of our proof. By 8 and 8" there exists a sequence of homomorphisms $t_1 \in Hom(Ln^{(1)}, Ln^{(1)}), t_2 \in Hom(Ln^{(2)}, Ln^{(2)}), \ldots$ such that each next one is an extension of the preceding one onto succeding variables satisfying 8 for $k = 1, 2, \ldots$. Let s be a substitution (i.e. $s \in Hom(Ln, Ln)$) obtained by joining those homomorphisms. We prove that s is the substitution we search for, i.e. $(*)$ holds.

Suppose that for some $a \in X$ it is false that $sa \in CPC$. Let n be such that $a \in Fr^{(n)}$. By 8 (for $k = n$ and $t = t_n$) we have $sa \in t_n(X \cap Fr^{(n)} \subseteq CPC$. Contradiction. Similarly we prove the second part of $(*)$. Q.E.D.

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