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Zdzisław Dywan

ON A CERTAIN METHOD OF PRODUCING LOGICAL MATRICES

The well-known method of producing logical matrices was presented by Jaśkowski in [1]. A theorem saying that the content of the product of matrices is equal to the intersection of contents of these matrices was given there. The proof of this theorem can be found in [2]. The construction of the product of matrices is simple. Another and simpler method of producing matrices will be given here. Moreover we will prove theorem on the representation of so called quasi-strongly finite consequences by finite matrices (see Corollary).

Let $\underline{L} = (L, Con)$ be an absolute free algebra generated from a denumerable set of generators (called propositional variables) by Con – a finite sequence of non-nullary operations called connectives. A mapping $C : 2^L \to 2^L$ is an operation of consequence of $\underline{L}$ iff for all X, Y $L - X \subseteq C(C(X)) \subseteq C(X \cup Y)$. Let C_1 and C_2 be consequence, then $C_1 \leq C_2$ iff for every $X \subseteq L - C_1(X) \subseteq C_2(X)$, and $C_1 = C_2$ iff $C_1 \leq C_2$ and $C_2 \leq C_1$. A consequence C is structural iff for every substitution $e : \underline{L} \to \underline{L}$ and every $X \subseteq L - eC(X) \subseteq C(eX)$. By the symbol Sb we denote a consequence (of $\underline{L}$) determined by the rule of substitution. By a matrix we understand a pair $M = (\underline{A}_M, D_M)$ where $\underline{A}_M = (A_M, Con_M)$ is an algebra similar to $\underline{L}$ and $D_M \subseteq A_M$. A relation between the formula $a \in L$ and the set of formulas $X \subseteq L$ such that

for every valuation $v : \underline{L} \to \underline{A}_M$ if $vX \subseteq D_M$ then $va \in D_M$

determines an operation of consequence and will be denoted by the symbol Cn_M.

Let $\underline{K}$ be a class of matrices whose algebras are similar to the language $\underline{L}$ – then $Cn_{\underline{K}} = inf\{Cn_M : M \in \underline{K}\}$, i.e. for every $X \subseteq L$, $Cn_{\underline{K}}(X) =$

$\bigcap_{M \in \underline{K}} Cn_M(X)$. If C is a structural consequence then $C \circ Sb$ ($\overline{C}$ for short) is a *quasi-structural consequence.* A class of matrices $\underline{K}$ is strongly (quasi-) adequate to (quasi-)structural consequence C iff $C = Cn_K$ ($C = \overline{Cn_K}$). If $\underline{K} = \{M\}$ then we say about strongly (quasi-) adequatness with respect to matrix M. A consequence C is (quasi-)strongly finite iff there is a finite set of finite matrices $\underline{K}$ strongly (quasi-)adequate to C.

Let $\underline{K}$ be a class of matrices (connected with $\underline{L}$) whose carriers of algebras are disjoint. $\underline{K}^* = (\underline{A}_{\underline{K}^*}, D_{\underline{K}^*})$ is a matrix connected with $\underline{L}$ and defined as follows:

$$\underline{A}_{\underline{K}^*} = (A_{\underline{K}^*}, Con_{\underline{K}^*}) \text{ is an algebra similar to } \underline{L}$$
$$\underline{A}_{\underline{K}^*} = (\bigcup_{M \in \underline{K}} A_M) \cup \{1_{\underline{K}}\} \text{ where } 1_{\underline{K}} \notin \bigcup_{M \in \underline{K}} A_M$$
$$D_{\underline{K}^*} = (\bigcup_{M \in \underline{K}} D_M) \cup \{1_{\underline{K}}\}$$

Let f be a k-ary connective form Con, then the respective operation $f_{\underline{K}^*}$ from $Con_{\underline{K}^*}$ is defined as follows:

$$f_{\underline{K}^*}(x_1, \ldots, x_k) = \begin{cases} f_M(x_1, \ldots, x_k) \text{ if } x_1, \ldots, x_k \in A_M \text{ for some } M \in \underline{K} \\ 1_{\underline{K}} \text{ otherwise} \end{cases}$$

EXAMPLES

I. Let $\underline{K} = \{M, N\}$ where the matrices M and N are defined as follows

$\to$	2	3	$\neg$
2	3	3	3
*3	2	3	2

$\to$	4	5	$\neg$
*4	5	5	5
5	4	5	4

where asterisk * denotes distinguished elements of matrix, $\to$ and $\neg$ are the only operations from Con.

Then the matrix $\underline{K}^*$ has the following form

$\to$	1	2	3	4	5	$\neg$
*1	1	1	1	1	1	1
2	1	3	3	1	1	3
*3	1	2	3	1	1	2
*4	1	1	1	5	5	5
5	1	1	1	4	5	4

$(1_{\underline{K}} = 1)$

II. Let $\underline{K} = \{M, N\}$ where matrices M and N are defined as follows

$\to$	2	3	4	5	$\neg$
2	5	5	5	5	5
3	4	5	5	5	4
4	3	4	5	5	3
*5	2	3	4	5	2

$\to$	6	7	8	$\neg$
6	8	8	8	8
7	6	8	8	6
*8	6	7	8	6

Then the matrix $\underline{K}^*$ has the following form

$\to$	1	2	3	4	5	6	7	8	$\neg$
*1	1	1	1	1	1	1	1	1	1
2	1	5	5	5	5	1	1	1	5
3	1	4	5	5	5	1	1	1	4
4	1	3	4	5	5	1	1	1	3
*5	1	2	3	4	5	1	1	1	2
6	1	1	1	1	1	8	8	8	8
7	1	1	1	1	1	6	8	8	6
*8	1	1	1	1	1	6	7	8	6

THEOREM (cf. [3], p. 77). *For every quasi-structural consequence there is a single matrix strongly quasi-adequate to it.*[1]

By [4] it is known that for every structural consequence C there is a class of matrices $\underline{K}$ such that $C = Cn_{\underline{K}}$. Hence for the proof of our theorem it is sufficient to prove that

$(*)$ $\overline{Cn_{\underline{K}}} = \overline{Cn_{\underline{K}^*}}$

($\leq$). Suppose that $a \notin \overline{Cn_{\underline{K}^*}}(X)$ $(a \in L,\ X \subseteq L)$. Since $\overline{Cn_{\underline{K}^*}}(X) = Cn_{\underline{K}^*}(Sb(X))$ then there is a valuation $v : \underline{L} \to \underline{A}_{\underline{K}^*}$ such that $vSb(X) \subseteq D_{\underline{K}^*}$ and $va \notin D_{\underline{K}^*}$. Let $e : \underline{L} \to \underline{L}$ be a substitution such that $ea = a$ and set eX contains only these variables which occur in a. Since $va \notin D_{\underline{K}^*}$ then $va \neq 1_{\underline{K}}$. Hence $va \in A_M$ for some $M \in \underline{K}$. Notice that for every set of formulas Y formed by means of the variables of the formula a, by virtue of the definition of $f_{\underline{K}^*}$, $vY \subseteq A_M$ holds. Hence $veX \subseteq A_M$. So $veX \subseteq A_M \cap vSb(X) \subseteq A_M \cap D_{\underline{K}^*}$ and $vea = va \in A_M - D_{\underline{K}^*} = A_M - D_M$. Hence $a \notin Cn_M(X)$. So we here proved that $Cn_{\underline{K}} \leq \overline{Cn_{\underline{K}^*}}$. Then $\overline{Cn_{\underline{K}}} \leq \overline{Cn_{\underline{K}^*}}$.

[1] The first proof of this result can be found in [3]. My proof gives an additional information on construction of this kind of matrix.

($\geq$). Notice that every matrix from $\underline{K}$ is a submatrix of matrix $\underline{K}^*$. Hence $Cn_{\underline{K}^*} \leq Cn_{\underline{K}}$. So $\overline{Cn_{\underline{K}^*}} \leq \overline{Cn_{\underline{K}}}$. Then the proof of our theorem is complete.

If $\underline{K}$ is a finite set of finite matrices then the equality ($*$) can be read as follows

COROLLARY. *Every quasi-strongly finite consequence can be determined by a single finite matrix.*

Propositional calculi are often built by intersection of sets of theses of other calculi. The method of producing of the matrices is sometimes used in constructing of it. We will demonstrate an alternative method. By ($*$) the following equality follows

($**$) $Cn_{\underline{K}}(\emptyset) = Cn_{\underline{K}^*}(\emptyset)$.

This equality shows us that $\underline{K}^*$ has the same properties as Jaśkowski's product. Moreover, we would notice that with respect to ($*$) in considerations of quasi-structural consequence the matrix $\underline{K}^*$ and the class of matrices $\underline{K}$ can be treated interchangeably. On the other hand, Jaśkowski's product has not this property – e.g. if M and N are matrices from Example I, then it can be proved that $\overline{Cn_{\{M,N\}}} \lneq \overline{Cn_{M \times N}}$.

References

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Department of Logic
The Catholic University of Lublin
Poland