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PROOF TABLEAU FORMULATIONS OF SOME FIRST-ORDER RELEVANT ORTHO-LOGICS

(Abstract)

In [6] proof tableau formulations were given of the implication/negation fragments of the important zero-order relevant logics E and R and the semi-relevant logic RM (for details of which see [1]). The main purpose of this paper then, is to extend results by giving proof tableau formulations of the *distribution-free fragments* of E, R and RM and of their first order extensions EQ, RQ and RMQ. Where X is one of these logics, we shall follow [13] in calling its distribution-free fragment OX – the 'O' standing for 'ortho' which is meant to signify the kinship of these logics to quantum logics or ortho-logics as they now tend to be known. Hence in what follows we shall refer to the logics OX simply as relevant ortho-logics.

Ortho-logics paradigmatically lack the principle of *distribution*, i.e.,

(1) $A\&(B \vee C) \to (A\&B) \vee (A\&C)$.

Therefore not surprisingly, axiomatizations of the logics OE, OR and ORM are obtained from the respective axiomatizations (given in [1]) of E, R and RM by just dropping (1) which is an axiom of these logics. Similarly, axiomatizations of the logics OEQ, ORQ and $ORMQ$ are obtained from the axiomatizations of EQ, RQ and RMQ (which can be extracted from [9]) by dropping (1) and the *confinement axiom*, i.e.

(2) $(\forall x)(A \vee B) \to \forall xA \vee B$ x not free in B.

Note that we augment the vocabulary of R, RQ, RM and RMQ by introducing the connectives $\circ$ (*fusion* or *intensional* conjunction) and $+$ (*fission* or *intensional disjunction*) which are defined in terms of negation and relevant implication as $\overline{A \to \overline{B}}$ and $\overline{A} \to B$ respectively. The existential quantifier is defined in EQ, RQ and RMQ in terms of the universal quantifier and negation in the usual way, and we consider these logics as augmented by the addition of a denumerable list of constants $a, b, c, a_1, \ldots$.

Here is not the place to argue the merits or otherwise of the relevant ortho-logics though this is done in [13]. But those logicians who are searching for logics with a simple proof theory, who are attracted by the logical insights reflected in the relevant logics and who have no great love for (1), would be advised to look into the merits of OEQ, ORQ and $ORMQ$. As we shall show, building on results to be found in [2], [4] and [8], they have straightforward Gentzenizations – the Gentzenization of ORQ in particular being essentially just the Gentzenization of the classical predicate calculus called $G1$ given in [3] without the structural rule of *weakening* or *thinning* as Kleene calls it. (Note that this Gentzenization of ORQ is *sequence-based* and it can be further simplified by dropping the structural rule of *permutation* – Kleene's *interchange* – resulting in a multiset-based Gentzenization of ORQ. For a discussion of the relationship between multiset and relevant implication see [5], [10] and [11]). Further, interpolation theorems of an appropriate kind can be proved for these logics (see [5]) while their propositional fragments are decidable (see [2], [4] and [8]) – these decision procedures having been automated (see [7]). However to the best of our knowledge, no-one has yet given an *interesting* semantics for these logics, though of course a semantics can be provided for them by using the methods of [12].

In what follows some familiarity with the details of [6] is assumed. For any one of the logics X, we let TX be its proof tableau formulation. With each system TX we associate (i) a set of rules for constructing proof tableaux and (ii) a set of *global requirements* which TX-proof tableau must obey. For all TX-proof tableaux τ and multisets α, α is said to be TX-refutable, in symbols $\alpha \vdash_{TX}$, iff (i) τ begins with α and (ii) τ satisfies the TX-global requirements. Hence $\vdash_{TX} A$ iff $\overline{A} \vdash_{TX}$.

We now give the set of rules and the global requirements that comprise the system $TORQ$. Note that rules are of two types – *connective rules* and *branch closure rules*.

CONNECTIVE RULES

Double Negation

$i \quad \overline{\overline{A}}$
.
.
.
$n \qquad\qquad \approx (i)$
$j \quad A$

Negated Implication

$i \quad \overline{A \rightarrow B}$
.
.
.
$n \qquad\qquad \tilde{\rightarrow}\,(i)$
$j \quad A$
$k \quad \overline{B}$

Implication

$i \quad A \rightarrow B$
.
.
.
$n \qquad\qquad \rightarrow (i)$
$j \quad \overline{A} \qquad\qquad k \quad B$ (branches)

Fusion

$i \quad A \circ B$
.
.
.
$n \qquad\qquad \circ\,(i)$
$j \quad A$
$k \quad B$

Negated Fission

$i \quad \overline{A + B}$
.
.
.
$n \qquad\qquad \tilde{+}\,(i)$
$j \quad \overline{A}$
$k \quad \overline{B}$

Negated Fusion

$i \quad \overline{A \circ B}$
.
.
.
$n \qquad\qquad \tilde{\circ}\,(i)$
$j \quad \overline{A} \qquad\qquad k \quad \overline{B}$ (branches)

Fission

$i \quad A + B$
.
.
.
$n \qquad\qquad +(i)$
$j \quad A \qquad\qquad k \quad B$ (branches)

Conjunction

$$\begin{array}{lll} i & A\&B & \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ n & & \&(i) \\ j & A & \end{array} \qquad\qquad \begin{array}{lll} i & A\&B & \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ n & & \&(i) \\ j & B & \end{array}$$

Negated Disjunction

$$\begin{array}{lll} i & \overline{A \vee B} & \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ n & & \tilde{\vee}\,(i) \\ j & \overline{A} & \end{array} \qquad\qquad \begin{array}{lll} i & \overline{A \vee B} & \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ n & & \tilde{\vee}\,(i) \\ j & \overline{B} & \end{array}$$

Disjunction

$$\begin{array}{lll} i & A \vee B & \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ n & & \vee\,(i, h_1, \ldots, h_n) \end{array}$$

$$\begin{array}{ll} j_1 \; A & j_2 \; B \\ \left.\begin{array}{l} k_1 \\ \cdot \\ \cdot \\ \cdot \\ k_n \end{array}\right] \alpha & \left.\begin{array}{l} m_1 \\ \cdot \\ \cdot \\ \cdot \\ m_n \end{array}\right] \alpha \end{array}$$

Negated Conjunction

$$\begin{array}{lll} i & \overline{A\&B} & \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ n & & \tilde{\&}\,(i, h_1, \ldots, h_n) \end{array}$$

$$\begin{array}{ll} j_1 \; \overline{A} & j_2 \; \overline{B} \\ \left.\begin{array}{l} k_1 \\ \cdot \\ \cdot \\ \cdot \\ k_n \end{array}\right] \alpha & \left.\begin{array}{l} m_1 \\ \cdot \\ \cdot \\ \cdot \\ m_n \end{array}\right] \alpha \end{array}$$

where α is an n-membered multiset of formulas assigned to the nodes k_1 to k_n and m_1 to m_n which contains all those *unused* formulas occurring at nodes $h_1, \ldots, h_m \leq n$ and which *may* also contain at most one instance of each of the *used* formulas occurring at nodes $h_{m+1}, \ldots, h_n \leq n$.

Universal Quantifier

i	$\forall x A(x)$	
.		
.		
.		
n		$\forall(i)$
j	$A(a)$	

Negated Existential Quantifier

i	$\sim\exists x A(x)$	
.		
.		
.		
n		$\tilde{\exists}\,(i)$
j	$\sim A(a)$	

where a is a constant.

Negated Universal Quantifier

i	$\sim\forall x A(x)$	
.		
.		
.		
n		$\tilde{\forall}\,(i)$
j	$\sim A(a)$	

Existential Quantifier

i	$\exists x A(x)$	
.		
.		
.		
n		$\exists(i)$
j	$A(a)$	

where a is a constant that does not occur in any formula at a node $h \leq n$.

BRANCH CLOSURE RULE

i	A	
.		
.		
.		
j	$\overline{A}$	$Cl(i,j)$

i	$\overline{A}$	
.		
.		
.		
j	A	$Cl(i,j)$

GLOBAL REQUIREMENTS

Closure Requirement

If is a $TORQ$-proof tableau, then τ satisfies the closure requirement iff every branch of τ is closed.

Use Requirement

If τ is a $TORQ$-proof tableau, then τ satisfies the use requirement iff each formula at a node in τ has a rule applied to it at least once.

The system TOE and $TORM$ easily obtained from this system by modifying the rule of negated implication and imposing a new global requirement, called the *barrier requirement*, in the case of the former and by simply generalizing the branch rule in the case of the latter. Details of those modifications can be extracted from [6]. We are the able to prove the main theorem of this paper.

THEOREM. *For all formulas* $A, \vdash_X A$ *iff* $\vdash_{TX} A$.

As in [6] this proof proceeds indirectly by firstly proving TX equivalent to its respective Gentzen system, from which the theorem follows immediately.

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