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AN ELEMENTARY METHOD OF DETERMINING THE DEGREE OF COMPLETENESS OF n-VALUED ŁUKASIEWICZ PROPOSITIONAL CALCULUS

Abstract

This paper was read at the seminar of the Chair of Logic of Jagiellonian University, October 26,1983.

The method of calculating the degree of completeness presented below differs from methods discussed in [1] and [2] and is explicitly based on elementary properties of Łukasiewiczian valuations and reveals the character of possible enlargements of given n-valued Łukasiewicz logic. Above all it seems to be especially convenient for didactic purposes.

C, N – implication and negation (in Polish notation)
α, β, γ – formulas (wffs) of implicational-negational n-valued Łukasiewicz calculus
FOR – the set of all formulas of implicational-negational n-valued Łukasiewicz calculus
v – an arbitrary Łukasiewicz valuation (n-valued)
$Sb(X)$ – the set of all substitutions of formulas from X
NWD – Minimal common divisor
$d(n-1)$ – number of all divisors of $n-1$ (counted with 1 and $n-1$)

$C(X) = \{\alpha \in FOR : \ \forall v(v(X) = \{1\} \Rightarrow v(\alpha) = 1\}$
$V(m) = \{i \in \{0, 1, \ldots, n-1\} : \ \exists k \ (i = k \cdot NWD(m, n-1))\}$ for $m > 0$.

We have always $0 \in V(m)$ and $n-1 \in V(m)$.

v_m – an Łukasiewicz valuation such that $v_m(FOR) \subset V(m)$.

LEMMA 1. $V(m) = V(NWD(m, n-1))$.

Hence the number of different sets $V(m)$ equals $d(n-1)$.

LEMMA 2. $v(\alpha), v(\beta) \in V(m) \Rightarrow v(C\alpha\beta), v(N\alpha) \in V(m)$. $E(V(m)) = \{\alpha \in FOR : \forall v_m(v_m(\alpha) = n-1)\}$.

LEMMA 3.
a) $V(m) \subset B(k) \Rightarrow E(V(k)) \subset E(V(m))$
b) $V(m) \neq V(k) \Rightarrow E(V(k)) \neq E(V(m))$,

It is obvious that $E(V(m)) \supset Ł_n$.

COROLLARY 1. *There are* $d(n-1)$ *different sets* $E(V(m))$.

This is an easy consequence of Lemmas 1 and 3.

LEMMA 4. $E(V(m)) = C(Sb(E(V(m))))$.

Trivially from Lemma 2.

$$|\alpha| = \begin{cases} min\{k \in \{0,1,\ldots,n-1\} : \alpha \in E(V(k))\} & \text{iff} \quad \exists m : \alpha \in E(V(m)) \quad 0 < m \leq n-1 \\ 0 & \text{iff} \quad \forall m : \alpha \notin E(V(m)) \end{cases}$$

LEMMA 5.
a) $|\alpha| = 0 \Rightarrow C(Sb(\alpha)) = FOR$
b) $|\alpha| > 0 \Rightarrow C(Sb(\alpha)) = E(V(|\alpha|))$.

Ad a). Let $|\alpha| = 0$. There exists $v \in V(n-1)$ such that $v(\alpha) = 0$. Now we define function e:

$$e(p) = \begin{cases} Cpp & v(p) = n-1 \\ & \text{iff} \\ NCpp & v(p) = 0 \end{cases}$$

Formula $h^e(\alpha)$ is a countertautology. As $h^e(\alpha) \in Sb(\alpha)$, we have $C(Sb(\alpha)) = FOR$.

Ad b). From the definition of $|\alpha|$ we have $\alpha \in E(V(|\alpha|))$. As C and Sb are monotonic, taking in account Lemma 4, we have that $C(Sb(\alpha)) \subset E(V(|\alpha|))$. Let $\beta \in E(V(|\alpha|))$ and $\beta \notin C(Sb(\alpha))$. There exists v such that v satisfies $Sb(\alpha)$ and $v(\beta) \neq n-1$. But for v satisfying $Sb(\alpha)$ we have $v \in V(|\alpha|)$, and this easily leads to the contradiction with our assumption, following which we must have $v(\beta) = n-1$.

Corollary 2. *The degree of completeness for L_n equals $d(n-1)+1$.*

It is an obvious consequence of Corollary 1 and Lemma 5.

References

[1] A. Rose, *The degree of completeness of m-valued Łukasiewicz calculus*, **the Journal of the London Mathematical Society** 27 (1952), pp. 92–102, 28 (1953), pp. 176–184.

[2] M. Tokarz, *Invariant systems of Łukasiewicz calculi*, **Zeitschrift für Mathematische Logik und Grundlagen der Mathematik** 20 (1974), pp. 221–228.

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