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THE SUFFICIENT AND NECESSARY CONDITION FOR TARSKI'S PROPERTY IN LINDENBUM'S EXTENSIONS

Abstract

The fall text of this paper will appear in Zeitschrift für mathematische Logik und Grundlagen der Mathematik.

0. The problem considered in this paper is connected with the well-known Tarski's theorem and concerns those systems which have only one Lindenbaum's extension.

I. Let S be the set of formulas formed in the usual manner by means of the infinite set of propositional variables (At) and the connectives from the set $\triangle$. Let N, K, A, C, E denote negation, conjunction, disjunction, implication and equivalence, respectively. We assume that $C \in \triangle \subseteq \{C, A, K, N, E\}$. $At(\alpha)$ ($\alpha \in S^{\triangle}$) denotes the set of all prepositional variables occurring in α.

The couple $\langle R, X\rangle$ is called a system of propositional calculus, whenever $X \subseteq S^{\triangle}$ and R is a set of primitive rules. For every $X \subseteq S^{\triangle}$, $Cn(R, X)$ is the smallest subset of $S^{\triangle}$ containing X and closed under the rules of R. By the symbol h^e we denote the extension of the function $e : At \to S^{\triangle}$ to the endomorphism. R_{o^*} denotes the set $\{r_o, r_*\}$ (r_o – modus ponens, r_* – substitution rule). $Sb_Z(X)$ denotes the set of all sentences which are obtained by replacing all variables in the sentence of the set X by sentences of the set Z. By R_{o^*} – systems we mean systems based on modus ponens and substitution rule and by R_o-systems we mean systems based on modus ponens. In this paper we will consider only R_{o^*}-systems and R_o-systems. Hence for the simplicity of the denotation we write $Cn(X)$ instead of $Cn(R_{o^*}, X)$ and $Cn_o(X)$ instead of $Cn(R_o, X)$.

Definition 1. $L(Cn(R,X)) = \{Y \subseteq S^{\triangle} : Cn(R,X) \subseteq Y = Cn(R,Y) \neq S^{\triangle} \wedge \forall_{\alpha \in S^{\triangle} - Y}\ Cn(R, Y \cup \{\alpha\}) = S^{\triangle}\}$

Definition 2. If $\psi \notin Cn(R,X)$ than $L^{\psi}(Cn(R,X)) = \{Y \subseteq S^{\triangle} : \psi \notin Y \wedge Cn(R,X) \subseteq Y = Cn(R,Y) \neq S^{\triangle} \wedge \forall_{\varphi \in S^{\triangle} - Y}\ \psi \in Cn(R, Y \cup \{\varphi\})\}$

Definition 3. $S_p^{\triangle} = \{\alpha \in S^{\triangle} : At(\alpha) = p\}$.

Notice that $L(Cn(R,X))$ is the set of all Lindenbaum's extensions of the set $Cn(R,X)$, while $L^{\psi}(Cn(R,X))$ is the set of all Lindenbaum-Asser's extensions of the set $Cn(R,X)$ for $\psi \in S^{\triangle} - Cn(R,X)$ (cf. [1]).

II. In this chapter at first we introduce the notion of the extensive-completeness of system $\langle R, X\rangle (\langle R, X\rangle) \in ECpl^{Z})$ as follows:

Definition 4. $\langle R, X\rangle \in ECpl^{Z}$ iff $\forall_{Y \in L(Cn(R,X))}\ \forall_{\alpha \in Z}\ \forall_{\beta \in Z - Y}\ \alpha \in Cn(R,X) \vee Cn(R, X \cup \{\alpha\}) = S^{\triangle} \vee Cn(R, X \cup \{C\alpha\beta\}) = S^{\triangle}$, where $Z \subseteq S^{\triangle}$ and $Cn(R,X) \subseteq S^{\triangle}$

For any $X \subseteq S^{\triangle}$ such that $Cn(R_{o^*}, X) \neq S^{\triangle}$ we define the sequence of the sets X_n $(n \in \mathcal{N} \cup \{0\})$ as follows:

$$X_o = Cn(X) \cap S_p^{\triangle},$$
$$X_{n+1} = \{\varphi \in X_n : \exists_{e: At \to S_p^{\triangle} - \{p\}}\ \exists_{\psi \in X_n}\ \varphi = h^e(\psi)\}$$

Besides, we define the sequence of the sets S_m $(m \in \mathcal{N} \cup \{0\})$:

$$S_o = S_p^{\triangle}$$
$$S_{m+1}^{\triangle} = S_m^{\triangle} - \{\varphi \in\ S_m : \sim \exists_{e: At \to S_p^{\triangle} - \{p\}} \exists_{\psi \in\ S_m^{\triangle} - \{p\}} \varphi = h^e(\psi)\}$$

Theorem 1. $\overline{\overline{L(Cn(X))}} = 1$ *iff* $\exists_{n,m \in \mathcal{N} \cup \{0\}} \langle R_{o^*}, X_n\rangle \in ECpl^{S_m^{\triangle}}$, *where* $Cn(X) \subsetneq S^{\triangle} (Z \subsetneq Y$ *iff* $Z \subseteq Y$ *and* $Z \neq Y)$.

Using Theorem 1 one can prove the following

Corollary. $\overline{\overline{L(Cn(X))}} = 1$ *iff* $\exists_{Z \subseteq S^{\triangle}} [Cn(X \cup Sb_Z(S^{\triangle})) = S^{\triangle} \wedge \langle R_{o^*}, X\rangle \in ECpl^{Sb_Z(S^{\triangle})}]$, *where* $Cn(X) \subsetneq S^{\triangle}$ *(cf. [2] and [3]).*

Next on the grounds of the R_o-systems we have the following

Theorem 2. $\overline{\overline{L(Cn_o(X))}} = 1$ *iff* $\langle R_o, X\rangle \in ECpl^{S^{\triangle}}$, *where* $Cn_o(X) \subsetneq S^{\triangle}$.

At least we establish these theorems which concern univocality of the Lindenbaum-Asser's extensions.

THEOREM 3. $\overline{\overline{L^{\psi}(Cn(X))}} = 1$ *iff* $\exists_{Y \in L^{\psi}(Cn(X))} \forall_{\alpha \in S^{\triangle}} \forall_{\beta \in S^{\triangle} - Y} \psi \in Cn(X \cup \{\alpha\}) \vee \psi \in Cn(X \cup \{C\alpha\beta\})$.

THEOREM 4. $\overline{\overline{L^{\psi}(Cn_o(X))}} = 1$ *iff* $\exists_{Y \in L^{\psi}(Cn_o(X))} \forall_{\alpha \in S^{\triangle}} \forall_{\beta \in S^{\triangle} - Y} \psi \in Cn_o(X \cup \{\alpha\}) \vee \psi \in Cn_o(X \cup \{C\alpha\beta\})$.

References

[1] G. Asser, **Einfürung in die mathematische Logic**, Tail I, Lepipzig, 1959.

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