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TOPOLOGICAL DUALITY FOR NELSON ALGEBRAS AND ITS APPLICATIONS (abstract)

Some results of this paper were presented at the VII-th Autumn Logical School held by the Section of Logic Polish Academy of Sciences, Podklasztorze (Poland), 16-25 November, 1983.

A *Nelson algebra* (= N-lattice or quasi-pseudo-Boolean algebra) $(A, \vee, \wedge, \rightarrow, \neg, \sim, 0, 1)$ is an algebra of the type $< 2, 2, 2, 1, 1, 0, 0 >$ which satisfies some appropriate axioms (for details see [7]). These axioms imply that the relation $\approx$ on A defined by: $a \approx b$ if and only if $a \rightarrow b = 1$ and $b \rightarrow a = 1$, is a congruence relation on $(A, \vee, \wedge, \rightarrow, \neg, 0, 1)$, and the quotient algebra $A^h = (A, \vee, \wedge, \rightarrow, \neg, 0, 1)/ \approx$ is a Heyting algebra. For a given Heyting algebra B there always exists a Nelson algebra A such that A^h is isomorphic to B: the Fidel-Vakarelov construction of the Nelson algebra $N(B)$ (see e.g. [8]) yields an example of such an algebra.

In this paper we describe all Nelson algebras A whose A^h's are isomorphic to a given Heyting algebra B; and next we consider some problems, related with this description, concerning equational and quasiequational subclasses (= subvarieties and subquasivarieties) of the class $\underline{N}$ of all Nelson algebras. Our description is obtained by an application of the topological duality theory of Priestley for bounded distributive lattices ([3], [4], [5]).

A partially ordered topological space x is said to be *totally order disconnected* if for all $x, y \in X$ with $x \not\leq y$ there exists a clopen increasing set U such that $x \in U$ and $y \notin U$. X is said to be a *h-space* if it is a compact totally order disconnected space such that for every open subset U of $X, (U]$ (= the smallest decreasing subset of X containing U) is open. A continuous order-preserving map f between h-spaces X and Y is defined to be a *h-space morphism* if for every $x \in X$, $f([x)) = [f(x))$.

PROPOSITION 1. (folklore, cf. [5]) *The category of Heyting algebras with Heyting algebra homomorphisms is dually equivalent to the category of h-spaces with h-space morphisms.*

Let X br a compact totally order disconnected space, an let g be an involutional (i.e. $g \circ g = id$) order- reversing homomorphism on X. Denote by $X^-(X^+)$ the set of all elements x in X such that $x \leq g(x)$ $(g(x) \leq x)$. (X, g) is said to be an *n-space* if the following holds:

(1) $X = X^- \cup X^+$,
(2) for all $x, y \in X^-$, if $x \leq g(y)$ then there exists $z \in X^-$ such that $x \leq z \leq g(x)$ and $y \leq z \leq g(y)$,
(3) for every clopen increasing subsets U and V of X, $(U \cap g(U) \backslash V]$ is clopen.

A continuous order-preserving map f between n-spaces (X_1, g_1) and (X_2, g_2) is defined to be an *n-space morphism* if $f \circ g_1 = g_2 \circ f$ and for all $x \in X_1^-$, $f([x) \cap X_1^-) = [f(x)) \cap X_2^-$.

PROPOSITION 2. *The category of Nelson algebras with Nelson algebra homomorphisms is dually equivalent to the category n-spaces with n-space morphisms.*

h-space morphisms and n-space morphisms which are homomorphisms will be called *h-isomorphisms* and *n-isomorphisms*, respectively.

If (X, g) is an n-space then the order subspace X^- (with induced topology and order) is a h-space. Moreover, if (X, g) is a dual n-space of a Nelson algebra A then X^- is, up to h-isomorphism, a dual h-space of the algebra A^h. So, to obtain a characterization of Nelson algebras A whose A^h's are isomorphic to a given Heyting algebra B it suffices to describe all n-spaces (X, g) whose order subspace X^- coincide with a dual h-space of the algebra B.

Let X and Y be disjoint, homomorphic and dually order isomorphic partially ordered spaces; and let the map $f: X \to Y$ establish the required order-reversing homomorphism. Furthermore, let S be an arbitrary subset of $Max(X)$ (= the set of all maximal elements in X). Define a topology $\mathcal{T}$, a relation $\leq$ and a map g on the set $X \cup f(X \backslash S)$ as follows:

(4) $Z \in \mathcal{T}$ if and only if $j^{-1}(Z)$ is open in X and $k^{-1}(Z)$ is open in Y, where the maps $j : X \to X \cup f(X \backslash S)$ and $k : Y \to X \cup f(X \backslash S)$ are defined by
$j(x) = x$ and $k(y) = \begin{cases} f^{-1}(y); & \text{if } y \in f(S) \\ y; & \text{otherwise,} \end{cases}$

(5) $\leq = \leq_X \cup \leq_{k(Y)} \cup (\leq_X \circ \varrho \circ \leq_{k(Y)})$, where $\leq_{k(Y)}$ is a partial order on $k(Y)$ induced from Y by k, and $\varrho = \{(x, k(f(x))); x \in X\}$, and

(6) $g(x) = \begin{cases} k(f(x)); & \text{if } x \in X \\ f^{-1}(x); & \text{otherwise.} \end{cases}$

Then $(x \cup f(X \backslash S), \mathcal{T}, \leq)$ is a partially ordered topological space. In the sequel we shall assume that the space Y and the map f are constructed in some canonical way, uniform for all spaces X; and hence this space will be denoted briefly by $X \nearrow S$.

THEOREM 3. *(i) If X is a h-space and the subset S of $Max(X)$ is closed in X (or equivalently closed in $Max(X)$) then $(X \nearrow S, g)$ is an n-space such that the order subspaces $(X \nearrow S)^-$ and $(X \nearrow S)^- \cap (X \nearrow S)^+$ coincide with X and S, respectively.*

(ii) Each n-space (Z, g) is n-isomorphic to the n-space $(X \nearrow S, g)$ for some h-space X and some closed subset S of $MaxX$.

Let B be a Heyting algebra and let X be its dual h-space. Then a given Nelson algebra A is isomorphic to $N(B)$ if and only if its dual n-space is n-isonorphic to $(X \nearrow \emptyset, g)$. On the other hand, since $Max(X)$ is closed in X, there exist Nelson algebras whose dual n-spaces are n-isomorphic to $(X \nearrow Max(X), g)$ (obviously, all of then are isomorphic). Select a one of such an algebra and denote it by $N_m(B)$.

The class $\underline{N}_m$ of all Nelson algebras whose dual n-spaces (X, g) satisfy $X^- \cap X^+ = Max(X^-)$ (each member of $\underline{N}_m$ is isomorphic to $N_m(B)$ for some Heyting algebra B) forms a variety which is, relative to $\underline{N}$, defined by an equation $(x \to\sim x) \wedge (\sim x \to x) = x \wedge \sim x$ (or equivalently by $((x \to\sim x) \wedge (\sim x \to x)) \to x = 1$. Moreover, the pair $(HSP(\underline{3}, \underline{N}_m)$ ($\underline{3}$ is a unique three element Nelson algebra) splits the lattice $\Omega(\underline{N})$ of all subvarieties of $\underline{N}$: hence $\Omega(\underline{N})$ is a disjoint union of two intervals $[HSP(\underline{3}))$ and $(\underline{N}_m]$.

Let $\Omega(\underline{H})$ be a lattice of all subvarieties of the variety $\underline{H}$ of all Heyting algebras. Define $\tau : \Omega(\underline{N}) \to \Omega(\underline{H})$ and $\sigma, \sigma_m : \Omega(\underline{H}) \to \Omega(\underline{N})$ as follows:

(7) $\tau(\underline{K}) = I(\underline{K}^h)$, for every $\underline{K} \in \Omega(\underline{N})$,
(8) $\sigma(\underline{K}) = IS(N(\underline{K}))$ and
(9) $\sigma(\underline{K}) = I(N_m(\underline{K}))$, for every $\underline{K} \in \Omega(\underline{H})$.

THEOREM 4. *(i) Each of τ, σ and σ_m is a complete lattice homomorphism.*
(ii) Both $\tau \circ \sigma$ and $\tau \circ \sigma_m$ are identities on $\Omega(\underline{H})$.
(iii) The image of $\Omega(\underline{H})$ under σ_m coincides with tho interval $(\underline{N}_m]$ of $\Omega(\underline{N})$.

COROLLARY 5. *The lattice $\Omega(\underline{H})$ is a retract of the lattice $\Omega(\underline{N})$ in the category of complete and distributive lattices with complete lattice homomorphisms; it is isomorphic to tho interval sublattice $(\underline{N}_m]$ of $\Omega(\underline{N})$.*

An analogous theorem to Theorem 4 holds also for the lattices $\Lambda(\underline{N})$ and $\Lambda(\underline{H})$ subquasivarieties of $\underline{N}$ and $\underline{H}$, respectively. It follows immediately from the fact that each of the operators $(.)^h, N$ and N_m preserves reduced products.

All presented above facts have an obvious logical interpretation. Moreover, we have the following

THEOREM 5. *The $\{\sim\}$-free fragment of the prepositional logic with strong negation determined by a variety $\underline{K} \in \Omega(\underline{N})$ coincides with the intuitionistic logic if and only if $\underline{N}_m \leq \underline{K}$ in the lattice $\Omega(\underline{N})$.*

This result together with an observation that the interval it $[\underline{N}_m)$ has a cardinality $2^{\aleph_0}$ (in fact, the lattice $\Omega(\underline{H})$ can be completely embedded into $[\underline{N}_m)$) imply the following

COROLLARY 6. *The set of all prepositional logics which are axiomatic extensions of the constructive logic with strong negation such that their $\{\sim\}$-free fragments coincide with the intuitionistic logic has a cardinality $2^{\aleph_0}$ and it contains the smallest logic, that is, the logic axiomatizated by a single formula $(p \leftrightarrow \sim p) \to p$.*

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