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CONJUNCTION WITHOUT CONDITIONS IN ILLATIVE COMBINATORY LOGIC

In [3] the prepositional connectives were defined in terms of the combinators K and S and the illative obs Ξ and H (ΞXY can be interpreted as "YV holds for all V such that XV holds" and HX can be interpreted as "X is a proposition"). Given an elimination rate for Ξ and introduction rules for H and Ξ, all the standard intuitionistic propositional calculus results could be proved provided the variables were restricted to H.

The intuition behind the particular introduction rule for Ξ of [2], that was used is [3], came from a three valued truth table for implication, the values of which were T, F and N. (N can be read as "nonsignificant" or "not T nor F"). There are in fact 4 different truth tables for implication that fit the rules for implication derived from those for Ξ. From these 4 different tables for conjunction ($\wedge$) and two for disjunction ($\vee$) (as well as one for negation and one for H) can be derived (see [4]).

The introduction and elimination rules for the connectives derived from the postulates for Ξ and H, where, with some exceptions, the most general ones that would fit all the truth tables. The exceptions were the elimination rules for $\wedge$ and $\vee$ which came out as:

$$HX, XY, \wedge HY \vdash X$$

$$HX, XY, \wedge XY \vdash Y$$

and $XZ,\ \vee XY, X \supset Z, Y \supset Z \vdash Z$

Even with some extra axioms connecting Ξ and H that were suggested in [5], it was only possible to prove the above with one of HX or XY removed. According to the truth tables (given below) none of HX, HY or HZ should be needed.

		Y		
	$X \supset Y$	T	F	N
X	T	T	F	N
	F	T	T	y
	N	x	N	N

		Y		
	$\wedge XY$	T	F	N
X	T	T	F	N
	F	F	F	b
	N	N	a	N

		Y		
	$\vee XY$	T	F	N
X	T	T	T	c
	F	T	F	N
	N	c	N	N

(1)	$x = y = N$	$a = b = N$	$c = N$
(2)	$x = T, y = N$	$a = F, b = N$	$c = N$
(3)	$x = N, y = T$	$a = N, b = F$	$c = N$
(4)	$x = y = T$	$a = b = F$	$c = T$

In this paper we show that when the introduction rule for Ξ of [2] is replaced by one of two more general forms (which in fact allow a limited form of higher order logic), a new definition of $\wedge$ can be written down which has unrestricted elimination rules as well as all other desirable properties of $\wedge$.

The problem of finding a new definition for $\vee$ that will satisfy the above eliminations rule as well as the other desirable properties of $\vee$ including

$$HX, HY \vdash H(\vee XY)$$

is still open.

A definition of $\wedge$ that gave unrestricted introduction and elimination rules was first given by Curry in [9]:

$$\wedge = \lambda x \lambda y.(u \supset_u vu) \supset_v (u \supset_u zvu) \supset_z z(Ky)x.$$

Carry's system however was inconsistent as it had an unrestricted deduction theorem (or introduction rule) for Ξ. Using a similar definition of $\wedge$ it was shown in [1] that the inconsistency of Kleene and Rosser of Church'a system (with K added), could also be based solely on the elimination rule for Ξ and a (somewhat restricted) deduction theorem.

One set of postulates that is adequate for all $\wedge$ theorems forms a subset of the postulates for the higher order logic of [6]. We have Rule Eq for equality, the Ξ elimination rule Rule Ξ, Rule H and only special cases of the introduction rule for Ξ and the postulates connecting H and Ξ.

Rule Eq	If	$X = Y$ then $X \vdash Y$
Rule Ξ		$\Xi XY, XV \vdash YV$
H		$X \vdash HX$
$DT\Xi$	If	$\triangle, XV \vdash YV,$

where V is an indeterminant not free in $\triangle$, X or Y, then

$$\triangle, LX \vdash \Xi XY$$

where $L = FHH$[1], $F(FHH)H, F(FHH)(FHH)$ or $F(FHH)(FHH))H$.

ΞH $\quad FUHX, FXHY \vdash H(\Xi XY)$

for $U = H$ or FHH,

FH If $U = H, FHH$ or $F(FHH)(FHH)$ then $\vdash FUHU$.

From these the rules for implication (P or $\supset$) can be derived as in [3].

RuleP	$X \supset Y, X \vdash Y$
RuleDTP	If $\triangle, X \vdash Y$ then $\triangle,\ HX \vdash X \supset Y$
RulePH	$HX, X \supset HY \vdash H(X \supset Y)$

Note that just as we writing $X \supset Y$ for the more formal PXY, we will usually write $Xu \supset_u Yu$ for ΞXY provided u is not in X or Y.

There al least two suitable forms for a definition of $\wedge$, one is an adaptation of Curry's definition:

$$\wedge = \lambda x \lambda y.F(FHH)(FHH)z \supset_z [(Hu \supset_u vu) \supset_v (u \supset_u zvu)] \supset z(Ky)x.$$

The one we use is:

Definition 1. $\wedge = \lambda x \lambda y.FH(FHH)z \supset_z [u \supset_u v \supset_v zuv] \supset zxy.$

We prove our main theorem for this definition of $\wedge$, but it can be proved equally well for the other.

Theorem 1.
(i) $\wedge XY \vdash X$
(ii) $\wedge XY \vdash Y$
(iii) $X, Y \vdash \wedge XY$

[1] F is given by: $F = \lambda x \lambda y \lambda z.\Xi x(Byz)$. $FXHY$ which is frequently used below, can therefore be written as $Xu \supset_u H(Yu)$.

(iv) $HX, HY \vdash X(\wedge XY)$

PROOF. (i) By $DT\Xi$, FH and the definition of F we have

$$\vdash FHHI, \tag{1}$$

so as $u, v \vdash Kuv$

we have by using $DT\Xi$ and (1) twice;

$$\vdash u \supset_u v \supset_v Kuv. \tag{2}$$

Also $Hx, Hy \vdash H(Kxy)$

so by $DT\Xi$ and FH twice: $\vdash FH(FHH)K$

by (2) and Definition 1

$$\wedge XY \vdash KXY$$

i.e. $\wedge XY \vdash X$

(ii) $u, v \vdash KIuv$

so by (1) and $DT\Xi$ $\vdash u \supset_u v \supset_v KIuv$ (3)

Also $Hx, Hy \vdash H(KIxy)$

so by FH and $DT\Xi$ $\vdash FH(FHH)(KI)$

so by FH and $DT\Xi$ $\vdash FH(FHH)(KI)$

by (3) and Definition 1

$$XY \vdash KIXY$$

i.e. $XY \vdash Y$

(iii) $FH(FHH)z, Hu, Hv \vdash H(zuv)$

so by Rule H,

$$FH(FHH)z,\ u,\ v \vdash H(zuv)$$

and by (1) and $DT\Xi$,

$$FH(FHH)z,\ u \vdash v \supset_v H(zuv).$$

Then by ΞH and (1),

$$FH(FHH)z,\ u \vdash H(v \supset_v zuv)$$

and by (1) and $DT\Xi$,

$$FH(FHH)z \vdash u \supset_u H(v \supset_v zuv).$$

so by ΞH and (1)

$$FH(FHH)z \vdash H(u \supset_u v \supset_v zuv) \tag{4}$$

Now $\quad X, Y,\ u \supset_u v \supset_v zuv \vdash zXY,$

so by DTP $\ X,\ Y,\ FH(FHH)z \vdash (u \supset_u v \supset_v zuv)zXY$

and by $DT\Xi$ and Definition 1, $\ X, Y \vdash\ \wedge XY$

(iv) $\quad HX, HY, FH(FHH)z \vdash H(uXY)$

so by (4) and DTP,

$$HX, HY, FH(FHH)z \vdash (u \supset_u v \supset_v zuv) \supset H(zXY)$$

Now by $DT\Xi$ $\quad HX, HT \vdash FH(FHH)z \supset_z H[(u \supset_u v \supset_v zuv) \supset zXY]$

so by $\Xi H, FH$ and Definition 1

$$HX, HY \vdash H(\wedge XY).$$

The weak consistency proof in [8] for the system of [7] has not so far been extended to the system used above. The consistency proof however does apply to the following extension of the system of [7]:

Rules Eq and H	as before
Rule Ξ	$UX, UV \vdash YV$
DT Ξ	If $\triangle, UV \vdash YV$ where V is an indeterminate not free in $\triangle$, or y then $\triangle \vdash \Xi UY$
ΞH	$FUHY \vdash H(\Xi UY)$

where $U \in \mathcal{U} = \{A, H, I, FAA, FAH, FAI, FHA, FIA, FHI, FIH, FHH, FII, FA(FAA), \ldots\}$.

Of course Rules P, DTP and PH are now no longer derivable from Ξ, $DT\Xi$ and ΞH, so they need to become rule of inference. On the other hand Axiom Scheme FH is now derivable.

It can easily be seen that Theorem 1 applies in this system even if only $U \in \{H, I, FH(FHH)\}$ is used. In fact the above proof can be simplified.

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