CONJUNCTION WITHOUT CONDITIONS IN ILLATIVE COMBINATORY LOGIC

In [3] the prepositional connectives were defined in terms of the combinators $K$ and $S$ and the illative obs $\Xi$ and $H$ ($\Xi XY$ can be interpreted as "$Y$ holds for all $V$ such that $XV$ holds" and $HX$ can be interpreted as "$X$ is a proposition"). Given an elimination rate for $\Xi$ and introduction rules for $H$ and $\Xi$, all the standard intuitionistic propositional calculus results could be proved provided the variables were restricted to $H$.

The intuition behind the particular introduction rule for $\Xi$ of [2], that was used is [3], came from a three valued truth table for implication, the values of which were $T$, $F$ and $N$. ($N$ can be read as “nonsignificant” or “not $T$ nor $F$”). There are in fact 4 different truth tables for implication that fit the rules for implication derived from those for $\Xi$. From these 4 different tables for conjunction ($\land$) and two for disjunction ($\lor$) (as well as one for negation and one for $H$) can be derived (see [4]).

The introduction and elimination rules for the connectives derived from the postulates for $\Xi$ and $H$, where, with some exceptions, the most general ones that would fit all the truth tables. The exceptions were the elimination rules for $\land$ and $\lor$ which came out as:

$$HX, XY, \land HY \vdash X$$

$$HX, XY, \land XY \vdash Y$$

and $XZ, \lor XY, X \lor Z, Y \lor Z \vdash Z$

Even with some extra axioms connecting $\Xi$ and $H$ that were suggested in [5], it was only possible to prove the above with one of $HX$ or $XY$ removed. According to the truth tables (given below) none of $HX, HY$ or $HZ$ should be needed.
In this paper we show that when the introduction rule for \( \Xi \) of [2] is replaced by one of two more general forms (which in fact allow a limited form of higher order logic), a new definition of \( \land \) can be written down which has unrestricted elimination rules as well as all other desirable properties of \( \land \).

The problem of finding a new definition for \( \lor \) that will satisfy the above eliminations rule as well as the other desirable properties of \( \lor \) including

\[
HX, HY \vdash H(\lor XY)
\]

is still open.

A definition of \( \land \) that gave unrestricted introduction and elimination rules was first given by Curry in [9]:

\[
\land = \lambda x \lambda y. (u \supset u)(v \supset (u \supset u) \supset (u \supset v) \supset (K y) x).
\]

Carry’s system however was inconsistent as it had an unrestricted deduction theorem (or introduction rule) for \( \Xi \). Using a similar definition of \( \land \) it was shown in [1] that the inconsistency of Kleene and Rosser of Church’s system (with \( K \) added), could also be based solely on the elimination rule for \( \Xi \) and a (somewhat restricted) deduction theorem.

One set of postulates that is adequate for all \( \land \) theorems forms a subset of the postulates for the higher order logic of [6]. We have Rule \( Eq \) for equality, the \( \Xi \) elimination rule Rule \( \Xi \), Rule \( H \) and only special cases of the introduction rule for \( \Xi \) and the postulates connecting \( H \) and \( \Xi \).
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Rule $Eq$  If $X = Y$ then $X \vdash Y$

Rule $\Xi$  $\Xi XY, XV \vdash YV$

$H$  \[ X \vdash HX \]

$DT\Xi$  If $\Delta, XV \vdash YV$,

where $V$ is an indeterminant not free in $\Delta$, $X$ or $Y$, then

$\Delta, LX \vdash \Xi XY$

where $L = FHH^1$, $F(FHH)H, F(FHH)(FHH)$ or $F(FHH)(FHH))H$.

$\Xi H$  $FUHX, FXHY \vdash H(\Xi XY)$

for $U = H$ or $FHH$,

$FH$  If $U = H, FHH$ or $F(FHH)(FHH)$ then $\vdash FUHU$.

From these the rules for implication ($P$ or $\supset$) can be derived as in [3].

Rule $P$  $X \supset Y, X \vdash Y$

Rule $DT P$  If $\Delta, X \vdash Y$ then $\Delta, HX \vdash X \supset Y$

Rule $PH$  $HX, X \supset HY \vdash H(X \supset Y)$

Note that just as we writing $X \supset Y$ for the more formal $PXY$, we will usually write $Xu \supset u$ for $\Xi XY$ provided $u$ is not in $X$ or $Y$.

There al least two suitable forms for a definition of $\land$, one is an adaptation of Curry’s definition:

$$\land = \lambda x\lambda y. F(FHH)(FHH)z \supset z \left[ (Hu \supset vu) \supset (u \supset vzu) \supset z(Ky)x \right].$$

The one we use is:

**Definition 1.** $\land = \lambda x\lambda y. FH(FHH)z \supset z \left[ u \supset v \supset zuv \supset zxy \right]$.

We prove our main theorem for this definition of $\land$, but it can be proved equally well for the other.

**Theorem 1.**

(i) $\land XY \vdash X$

(ii) $\land XY \vdash Y$

(iii) $X, Y \vdash \land XY$

\footnote{$F$ is given by: $F = \lambda x\lambda y\lambda z. \Xi x(Byz)$. $FXHY$ which is frequently used below, can therefore be written as $Xu \supset u H(Yu)$.}
(iv) $HX, HY \vdash X(\land XY)$

**Proof.** (i) By $DT\Xi$, $FH$ and the definition of $F$ we have

$$\vdash FHHI,$$  \hspace{1cm} (1)

so as $u, v \vdash Kuv$

we have by using $DT\Xi$ and (1) twice:

$$\vdash u \supset u \supset v Kuv.$$  \hspace{1cm} (2)

Also $Hx, Hy \vdash H(Kxy)$

so by $DT\Xi$ and $FH$ twice: $\vdash FH(FHH)K$

by (2) and Definition 1

$$\land XY \vdash KXY$$

i.e. $\land XY \vdash X$

(ii) $u, v \vdash KIuv$

so by (1) and $DT\Xi$ $\vdash u \supset u \supset v KIuv$

Also $Hx, Hy \vdash H(KIxy)$

so by $FH$ and $DT\Xi$ $\vdash FH(FHH)(KI)$

so by $FH$ and $DT\Xi$ $\vdash FH(FHH)(KI)$

by (3) and Definition 1

$$XY \vdash KIXY$$

i.e. $XY \vdash Y$

(iii) $FH(FHH)z, Hu, Hv \vdash H(zuv)$

so by Rule $H$,

$$FH(FHH)z, u, v \vdash H(zuv)$$

and by (1) and $DT\Xi$, 

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\[ FH(FHH)z, \ u \vdash v \supset H(zuv). \]

Then by \( \Xi H \) and (1),

\[ FH(FHH)z, \ u \vdash H(v \supset zuv) \]

and by (1) and \( DT\Xi \),

\[ FH(FHH)z \vdash u \supset H(v \supset zuv). \]

so by \( \Xi H \) and (1)

\[ FH(FHH)z \vdash H(u \supset v \supset zuv) \tag{4} \]

Now let \( X, Y, u \supset v \supset zuv \vdash zXY \),

so by \( DTP \ X, Y, FH(FHH)z \vdash (u \supset v \supset zuv)zXY \)

and by \( DT\Xi \) and Definition 1, \( X, Y \vdash \land XY \)

(iv) \( HX, HY, FH(FHH)z \vdash H(uXY) \)

so by (4) and \( DTP \),

\[ HX, HY, FH(FHH)z \vdash (u \supset v \supset zuv) \supset H(zXY) \]

Now by \( DT\Xi \)

\[ HX, HT \vdash FH(FHH)z \supset H[(u \supset v \supset zuv) \supset zXY] \]

so by \( \Xi H, FH \) and Definition 1

\[ HX, HY \vdash H(\land XY). \]

The weak consistency proof in [8] for the system of [7] has not so far been extended to the system used above. The consistency proof however does apply to the following extension of the system of [7]:

Rules Eq and H as before

Rule \( \Xi \)

\[ UX, UV \vdash YV \]

\[ DT\Xi \] If \( \triangle, UV \vdash YV \) where \( V \) is an indeterminate not free in \( \triangle \), or \( y \) then \( \triangle \vdash \Xi UY \)

\[ \Xi H \]

\[ FUHY \vdash H(\Xi UY) \]

where \( U \in \mathcal{U} = \{ A, H, I, FAA, FAH, FAI, FHA, FIA, FIA, FHI, FIH, FHH, FII, FA(FAA), \ldots \} \).
Of course Rules $P$, $DTP$ and $PH$ are now no longer derivable from $\Xi$, $DT\Xi$ and $\Xi H$, so they need to become rule of inference. On the other hand Axiom Scheme $FH$ is now derivable.

It can easily be seen that Theorem 1 applies in this system even if only $U \in \{H, I, FH(FHH)\}$ is used. In fact the above proof can be simplified.

References


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