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Bogusław Wolniewicz

A TOPOLOGY FOR LOGICAL SPACE

1. Algebra of Subsets

To generalize as in [7] the constructions of [2] and [5], let L be a non-degenerate join-semilattice with unit. With $A \cdot B = \{x \vee y \in L : x \in A, y \in B\}$ and $A^{\perp} = \{y \in L : x \vee y = 1 \text{ for all } x \in A\}$, the structure $(P(L), \cdot, \cup, ^{\perp}, L, \emptyset)$ is the *algebra of subsets* for L.

Let $\underline{R}$ be the maximal ideals of L. Interpreting L as the totality of *elementary situations*, and $1 \in L$ as the *impossible* one (cf. [1], [3], and [6]), the members of $\underline{R}$ will be called *realizations*, and $\underline{R}$ itself – the *logical space* associated with L. (Both the term – *logischer Raum* - and the idea derive from Wittgenstein's *Tractatus*.)

As it turn out, the *V-equivalence* relation defined as $A \sim_v B$ iff ($A \cap R = \emptyset$ iff $B \cap R = \emptyset$), for all $R \in \underline{R}$, is a congruence of the algebra of subsets $P(L)$, and the factor algebra $P(L)/v$ is a bounded distributive lattice.

In [7] we have considered the following compactness condition for L:

(C) $\forall_{A_i \in Fin}(A_i \subset A \Rightarrow A_i \cap R \neq \emptyset) \Rightarrow A \cap R \neq \emptyset$,

for any $R \in \underline{R}$, and with Fin being the finite subsets of L. As has been pointed out to us independently by Dr. J. Hawranek and Prof. A. Wroński, the main result of [7] – i.e. "Proposition 2" there – may be strengthened to the characterization:

(1) $P(L)/v$ is a Boolean algebra iff L satisfies condition (C).

2. Closed Subsets

Setting $V(A) = A^{\perp\perp}$ as in [4] we get a closure operation such that, for any $A, B \subset L$:

(2) If L satisfies (C), then $A \sim_v V(A)$, $V(A) = \bigcup /A/_v$, and $A \sim_v B$ iff $V(A) = V(B)$.

With $\underline{V} = \{A \subset L : \; A = V(A)\}$, and $A \dot{\cup} B = V(A \cup B)$, the structure $(\underline{V}, \cap, \dot{\cup}, ^{\perp}, L, \{1\})$ is the *algebra of closed subsets* for L. And we have:

(3) If L satisfies (C), then the algebra of closed subsets $\underline{V}$ is a complete Boolean algebra ordered by set inclusion and isomorphic to the factor algebra $P(L)/v$.

Setting yet $\underline{r}(A) = \{R \in \underline{R} : A \cap R \neq \emptyset\}$ we have also:

(4) If L satisfies (C), then the map $\underline{r} : \underline{V} \to P(R)$ is an embedding of $\underline{V}$ in the field of sets $P(\underline{R})$.

3. The Topology

Under condition (C) the logical space $\underline{R}$ may be given a fairly natural topology.

Take as open the sets of realizations which have the form $\underline{r}(V)$, with $V \in \underline{V}$; i.e., those constituting the range $\underline{r}/\underline{V}/$ of the map $\underline{r}$. The base for that topology are the open sets determined by the closed subsets of L which have the form $V(x) = V(\{x\})$, with $x \neq 1$. If, moreover, L is atomistic, then the sets of form $\underline{r}(V(x))$, where x is an atom of L, are a subbase for $\underline{r}/V/$.

As defined, $\underline{R}$ is a T_1-space. Moreover, since $\underline{r}(V)$ is open, $\underline{R} - \underline{r}(V)$ is closed. But $A^{\perp} = A^{\perp\perp\perp}$, so $V^{\perp} \in \underline{V}$. Thus the open sets coincide here with the closed ones. Consequently,

(5) If L satisfies (C), then $(\underline{R}, \underline{r}/\underline{V}/)$ is a zero-dimensional topological space.

4. Separation

Setting $r(x) = \{R \in \underline{R} : x \in R\}$, call elements $x, y \in L$ separable – for short: $Sep(x, y)$ – iff $r(x) \neq r(y)$. If $Sep(x, y)$ for any $x \neq y$ of L, then L is *separated*. With $N(x, y)$ iff $r(x) = r(y)$, the *inseparability* relation N is an equivalence on L, and since $r(x \vee y) = r(x) \cap r(y)$, it is also a congruence of it. Thus the partition L/N into blocks of inseparable elements is a homomorphic image of L.

For the atoms of $\underline{v}$, if any, we have the characterizations:

(6) The following conditions are mutually equivalent:
  (a) V is an atom of $\underline{V}$;
  (b) $V \neq \{1\}$, and $A \subset V$ implies $V(A) = V$, for any $A \neq \emptyset$, $\{1\}$;
  (c) $V \neq \{1\}$, and all members of $V - \{1\}$ are inseparable;
  (d) $\underline{r}(V) = \{R\}$, for some $R \in \underline{R}$.

However,

(7) If L satisfies (C), then $V(x)$ is an atom of V iff the block $/x/_N$ is a dual atom of the factor semi-lattice L/N.

So we obtain the result:

(8) If L satisfies (C), then the Boolean algebra $\underline{V}$ is atomic iff the factor semilattice L/N is dually atomic.

As corollaries of propositions (7) and (8) we have the following:

(9) If L satisfies (C), then $V(x) = V(y)$ iff $N(x, y)$.
(10) If L is separated and satisfies (C), then $\underline{V}$ is atomic iff L is dually atomic.
(11) If L satisfies (C), then the Boolean algebra $\underline{V}$ is isomorphic to the field of sets $P(\underline{R})$ iff L/N is dually atomic.
(12) If L/N is dually atomic, then $\underline{r}/\underline{V}/ = P(\underline{R})$, i.e. the topology defined on $\underline{R}$ is a discreet one.

References

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Institute of Philosophy
Warsaw University
Poland