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ON COMPLETE BUNDLES OF LOCALLY VALID IDENTITIES

Abstract

This is an abstract of the paper submitted to Algebra Universalis.

Some simple algebraic properties, described by universally quantified disjunctions of special identities, are established. Such sentences seem to be useful for an investigation on finite algebras and its products. These considerations are exemplified by results concerning distributive lattices.

By $P^{(\omega)}(t)$ the polynomial algebra of a finite type t is understood. $P(n)(t)$ is the n-ary polynomial algebra ([2]). No notational distinction is made between an algebra and its underlying set. Similarly by the same symbol is denoted each polynomial and its realization in a given algebra.

Definition 1. The symbol $\{p_1, p_2, \dots, p_n\} \rhd q$ denotes the sentence

$$\forall_{\overline{x}}\ [p_1(\overline{x}) \text{ or } p_2(\overline{x}) = q(\overline{x}) \text{ or} \dots \text{or } p_n(\overline{x}) = q(\overline{x})]$$

where $\overline{x} = (x_1, x_2, \dots, x_m)$ and $p_1, \dots, p_n, q \in P^{(m)}(t)$ for $m, n \in N$. Such sentences are called algebraic rules ([1]). Polynomials $p_1, p_2, \dots, p_n$ are premises of the rule $\{p_1, \dots, p_n\} \rhd q$ and q is its conclusion.

If $\{p_1, \dots, p_n\} \rhd q$ is true in an algebra A, then for every $\overline{a} \in A^m$ the set $\{p_1(\overline{a}), \dots, p_n(\overline{a})\}$ contains the one element set $\{q(\overline{a})\}$. In other words $p_1 = q$, $p_2 = q, \dots, p_n = q$ constitute a complete bundle of identities locally valid in A. In particular for $n = 1$ the rule $\{p\} \rhd q$ is in fact an equation and will be written simply as $p = q$.

For instance the rule

$$\{x_1 \wedge x_2, x_1 \wedge x_3, x_2 \wedge x_3\} \models x_1 \wedge x_2 \wedge x_3$$

is valid in $C_m \times C_n$, where C_k is the k-element chain treated as a lattice.

$Ar(A)$ denotes the set of all algebraic rules true in A and $Id(A)$ is the set of equations possessed by A. Always a convention is assumed that $Id(A) \subseteq Ar(A)$. Put $Ar(K) = \bigcap\{Ar(A) : A \in K\}$ for each class K of similar algebras.

DEFINITION 2. The rule $\{p_1, \ldots, p_n\} \models q$ is called proper in an algebra A if it is valid in A and $p_i = q \notin Id(A)$ for $i = 1, 2, \ldots, n$, where $p_1, \ldots, p_n, q \in P^{(m)}(t)$. This proper rule is called irreducidable in A if for each $i = 1, 2, \ldots, n$ there exists $\overline{a}_i \in A^m$ such that $p_i(\overline{a}_i = q(\overline{a}_i)$ and $p_j(\overline{a}_i) \neq q(\overline{a}_i)$ for $j \neq i$.

Clearly, if a rule is proper in A then eliminating its premises it is possible to obtain an irreducible rule in A.

Algebraic rules as positive sentences are preserved under the formation of homomorphic images and subalgebras. Hence $HS(K) \subseteq Mod(K)$ for each class K of algebras. Sometimes, as it will be shown, the converse inclusion is true.

One may see that algebraic rules are not stable under the formation of direct products In fact the following is valid.

PROPOSITION 1. *If $Ar(A^k) - Id(A) \neq \emptyset$, then $Ar(A^{k+1}) \subset Ar(A^k)$ (this is the proper inclusion), for every algebra A and each $k \in N$, where A^k denotes the k-th power of A.*

Sometimes from a rule in A^k it is possible to obtain a new one valid in A^{k+1}.

LEMMA 1. *If $\{p_1, \ldots, p_n\} \models q$ is true in C_2^k, where $p_1, \ldots, p_n, q$ are m-ary lattice polynomials, then $\{p_1 \wedge x_{m+1}, p_2 \wedge x_{m+1}, \ldots, p_n \wedge x_{m+1}, q\} \models q \wedge x_{m+1}$ and $\{p_1 \vee x_{m+1}, p_2 \vee x_{m+1}, \ldots, p_n \vee x_{m+1}, q\} \models q \vee x_{m+1}$ are valid in C_2^{k+1} (C_2 is the 2-element lattice).*

Observe that $\{x_1, x_2\} \models x_1 \wedge x_2$ is true in C_2 and false in C_2^2. Hence by Lemma 1 and Proposition 1 the next statement follows.

LEMMA 2. *$Ar(C_2^{k+1}) \subset Ar(C_2^k)$ for each $k \in N$.*

In order to state an interesting proposition let us define, for every lattice L, the lattices:

$L+a = <L \cup \{a\}, \wedge', \vee'>$ and $L+b = <L \cup \{b\}, \wedge', \vee'>$ where $a, b \notin L$, by

$x \vee' y = x \vee y, x \wedge' y = x \wedge y$ and $x \vee' a = x, x \wedge' a = a$, $y \vee' b = b$, $y \wedge' b = y$ for every $x, y \in L$.

THEOREM 1. *$Mod(Ar(C_2^k)) = HS(C_2^k)$ for each $k \in N$.*

However, not for every distributive lattice L, $Ar(L+a) \neq Ar(L) \neq Ar(L+b)$. For instance $Ar(C_3) = Ar(C_n)$ for $n \geq 3$ (see [1]). By the following general result it will be possible to construct another distributive lattices with this property.

PROPOSITION 2. *If $Ar(A_1) = Ar(B_1)$ and $Ar(A_2) = Ar(B_2)$, then $Ar(A_1 \times A_2) = Ar(B_1 \times B_2)$ for all similar algebras A_i, B_i, $i = 1, 2$.*

For instance $Ar(C_h \times C_k) = Ar(C_m \times C_n)$ for all natural numbers $h, k, m, n \geq 3$. On the other hand $Ar(C_2 \times C_3) \neq Ar(C_3 \times C_3)$ as it may be visible by the rule $\{z, z \wedge x, z \wedge y, x \vee y\} \models z \wedge (x \vee y)$ valid in $C_2 \times C_3$ only.

In general the following open problem seems worth to be stated:

Conjecture. If $Ar(A_1) \neq Ar(B_1)$ or $Ar(A_2) \neq Ar(B_2)$ then $Ar(A_1 \times A_2) \neq Ar(B_1 \times B_2)$ for algebras A_i, B_i belonging to the same variety.

Observe that the number of premises of any rule valid in A^k has a lower bound for each algebra A (compare Lemma 1).

PROPOSITION 3. *If $\{p_1, \ldots, p_n\} \models q$ is proper in A^k, then $n \geq k+1$ for every $k, n \in N$ and each algebra A.*

Finally let as state that if $F_K(n)$ denotes the n-generated K-free algebra and r is proper in it, then r properly depends on at least $n+1$ variables. Thus $Ar(F_K(X)) = Id(F_K(X))$ for card $(X) = \omega$.

References

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