Bulletin of the **S**ection of **L**ogic
Volume 13/2 (1984), pp. 92–95
reedition 2008 [original edition, pp. 92–98]

Zdzisław Dywan

ON LEMMON'S INTERPRETATIONS OF THE CONNECTIVE OF NECESSITY

The reason of writing this paper is the discovery of some gape and errors in Lemmon's paper [4], which again pose before us the problem of interpretation of the connective of necessity. We will show here that a gap which occurs in the consideration of $S5$ can be completed.

The term correctness under interpretation plays the key role in Lemmon's considerations. I think it will be best to quote Lemmon's own explanation at this point. "... interpretation must involve ... the assignation of words belonging to some language to the formal symbols, in such a way that formulae of the calculus are transformed into sentences of the language. ... There are conventionally understood ways of doing this; e.g., '... $\vee$ ...' becomes 'either ... or ...'. A set of such understood transformations I shall call an interpretational key. If interpretation into a natural language is envisaged, this key will very likely be exceedingly complex; think of specifying fully how brackets are to go over into punctation-marks (such as commas). Once this key ie supplied, it seems natural to define a formula of a calculus as correct if the sentences obtained from it under the key is such that any statement made by using them is true, and as incorrect if some sentence obtained from it under the key is such that some statement made by using it is false. The calculus may then be said to be correct if its theorems and non-theorems coincide respectively with the correct and incorrect formulae."

Let us go into the details. Lemmon considers four modal logics: $SO.5, M, S4, S5$. Let these symbols also denote the respective sets of theses of these logics. We focus our attention, for the present, on Lemmon's considerations connected with the calculus $S4$. He believes that with respect to a certain interpretation this calculus ia correct, showing that:

(i) $S4$-axioms are correct under that interpretation,

(ii) The primitive rules of $S4$ are acceptable under that interpretation,

(iii) The formula $Lp \vee LNLp$[1] (which added to $S4$-axioms gives the axiomatic of the next one of the four mentioned logics, i.e. $S5$-axiomatic) is incorrect under that interpretation.

The reader is now asked to compare the last sentence in the above quotation with (iii). If we denote by the symbol $S4^+$ the set of all formulas correct under his interpretation, then by showing (i)-(iii) we can say that the following disjunction is true:

$$S4^+ \not\subseteq S5 \text{ or } S4^+ - S5 \neq \emptyset$$

but not, as Lemmon wishes that $S4^+ = S4$.
It seems that he unconsciously assumed the following false assertion:

Every axiomatic extension of $S4$ contains $S5$

The quality $S4^+ = S4$ follows from this assertion and the above disjunction. Similar objection can be made when Lemmon considers the calculi $SO.5$ and M. In considering $S5$ Lemmon proves only (i) and (ii) (for $S5$ obviously) and wrongly states that under his interpretation $S5$ is a correct calculus.

There is also another very disturbing point in Lemmon's paper. Considering the calculus $SO.5$ he reads the connectives of necessity metalogically – 'it is tautologous (by truth table) that'; then he shows that the formula $L\ (Lp \supset p)$ is not acceptable under his interpretation and says: "though it may be a logical truth that what is tautologous is true, it is not a tautology that what is tautologous is true.". Note that by the same argumentation we would have to accept the formula $LLp \supset q$ (that something is a tautology is not a tautology, then if we assert the contrary anything will follow). So the logic obtained in such a way contains $SO.5$ but is not contained in $S5$. Let us further note that in this argumentation there is a certain misunderstanding. If the connective L is read – 'it is tautologous (by truth table) that', then the interpretation of the formula La, where a contains the connective L, is not false but unfeasible nonsense. The formula La can be interpreted only when a contains only classical connectives.

[1]The letters $p, q, \ldots$ will denote sentential variables; the letters $a, a_1, a_2, \ldots$ formulas of modal language; symbols $L, N, \supset, \vee$ respectively the connectives of: necessity, negation, implication, disjunction.

A similar objection can be levelled against Gödel when he considers the formula $L(Lp \supset p)$[2] in [3]. In the same paper he writes up the $S4$-axiomatic and states ad hoc that the connective of necessity can be reed as 'is provable'.

In spite of these objections, I think that one Lenmon's, or strictly speaking Carnap's (cf. 1 pp. 174-175), interpretations can be accepted. Understanding Lp as – 'it is analytically the case that p' Lemmon shows that $S5$-axioms are correct formulas under this interpretation and the primitive rules of $S5$ (substitution, modus ponens and necessitation – if $\vdash a$ then $\vdash La$) are also acceptable under this interpretation.
It still needs to be proved that under this interpretation all non-theses of $S5$ are incorrect. In [5] it is shown that we can reject all non-theses of $S5$ admitting the following

$$\frac{\dashv a \supset a_1, \ldots, \dashv a \supset a_k}{\dashv La \supset La_1 \vee \ldots \vee La_k} \quad R$$

where the formulas $a, a_1, \ldots, a_k$ do not contain the connective L, and the symbol $\dashv$ is understood as the symbol of rejection in $S5$. Lemmon has some trouble in understanding the term 'analytic' and he comments that the expression 'it is analytically the case that p' can be understood as follows – "it is the case that p, solely in virtue of the meanings of the words in the sentence used to make the statement that p.". From this explanation follows the following theorem

(1) There are infinitely many analytically false independent sentences, i.e. every complex sentence formed from them by means of classical connectives is analytically true iff it is a substitution of a classical thesis.

E.g. for $n = 3$ the following are such sentences: 'John sleeps', 'Some dog runs' and 'Every building has two floors'. It is obvious that without empirical verification are unable to decide about the logical values of these sentences, or of complex sentences formed from them by means of classical connectives, unless they are substitutions classical theses of counter-theses. I think that it is not necessary to show the truth of (1) for every natural

[2]Basing on the technique of arithmetization and treating L term defined in a certain manner, Gödel's considerations in the final part of [3] can be regarded as correct (see also [4] pp. 32–33).

number. By (1) and a certain theorem from [2] the acceptability of the rule R can be proved.

Finally one more question concerning the interpretation the connective L in $S4$. Lemmon proposes to read it as 'it informally provable in mathematics that'. Assume that the calculus obtained under this interpretation, i.e. $S4^+$ in our notation, is $S5$. If so, the rule R should be acceptable in $S4^+$. In our proof of acceptability of this rule (for $S5$) an essential part is played by theorem (1). It is a question whether such an assumption is satisfied under the above reading of L, i.e.:

There are infinitely many provably independent mathematical sentences, i.e. every complex sentence formed from them by means of classical connectives is provable iff it is a substitution of a classical thesis.

I think that the above hypothesis is true.

To sum up, the question of interpretation of the connective L in the logics $SO.5$. M and $S4$ are open. A different interpretation must be chosen for $SO.5$, while in the case of M and $S4$ one can tray to fill the gaps in Lemmon's considerations.

References

[1] R. Carnap, **Meaning and necessity, a study in semantics and modal logic**, Chicago, The University of Chicago Press, 1947.

[2] Z. Dywan, *On a certain strengthening of Gödel-Malcev theorem for the classical propositional calculus*, to appear.

[3] K. Gödel, *Eine interpretation der intuitionistischer Aussagenkalküls*, **Ergebnisse eines mathematisches Kolloquims**, Vol. 4 (1933), pp. 39–40.

[4] J. Lemmon, *Is there only correct system of modal logic?*, **Aristotelian Society Suppl.** 33 (1959), pp. 23–40.

[5] J. Słupecki and G. Bryll, *Proof of L-decidability of Lewis system* $S5$, **Studia Logica**, Vol. XXXII (1973), pp. 99–105.

Department of Logic
The Catholic University of Lublin
Poland