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CHARACTERIZATION OF PRIME NUMBERS BY ŁUKASIEWICZ'S MANY-VALUED LOGICS

In this note we construct finitely n-valued logic K_n such that it has tautologies iff $n-1$ is a prime number. Moreover, we prove that K_n with respect to functional properties is Łukasiewicz's n-valued logic $Ł_n$ when $n-1$ is a prime number. The proof in direct way shows the exceptional complexity of distributing prime numbers in natural series.

Let us recall the definition of Łukasiewicz's n-valued logic (cf. Łukasiewicz and Tarski [3]).

First, let $\mathcal{M}_n^{Ł} = < M_n, \sim, \rightarrow, \{n-1\} >$ where $n \in N$ and $n \geq 2$, be Łukasiewicz's n-valued matrix. That is, $M_n = \{0, 1...., n-1\}$ and $\sim x = n-1-x$, $x \rightarrow y = min(n-1, n-1-x+y)$ and $\{n-1\}$ is the set of designated elements of $\mathcal{M}_n^{Ł}$.

Second, to each algebra $< M_n, \sim, \rightarrow >$, $n \geq 2$, of the matrix $\mathcal{M}_n^{Ł}$ there corresponds an unique propositional language, SL say generated by a denumerable set of propositional variables $\{p, q, r, ...\}$ say, and the two connectives: $\neg$ (negation) and $\supset$ (implication).

Finally, we define Łukasiewicz's n-valued logic $Ł_n$ to be the set of all tautologies of the matrix $\mathcal{M}_n^{Ł}$, i.e. the set of all formulas α such that $v(\alpha) = n-1$ for each valuation v of SL into $\mathcal{M}_n^{Ł}$ where v is homomorphism from SL into $\mathcal{M}_n^{Ł}$.

We denote the set of all matrix functions from $Ł_n$ by $\mathcal{L}_n$. Let P_n be the set of all n-valued functions defined on the set $\{0, 1, ..., n-1\}$. Then the set of functions R is called *functionally precomplete* (in P_n) set if an addition to R of a function $f \notin R$ forms the set $\{R, f\}$ *functionally complete*, i.e. if $\{R, f\} = P_n$. In [1] Bochvar and Finn have proved the set of functions $\mathcal{L}_n$ is functionally precomplete in P_n iff $n-1$ is a prime number.

Now we define the matrix $\mathcal{M}_n^k$ in the following way:

$$\mathcal{M}_n^k = < M_n, \sim x, x \xrightarrow{k} y, \{n-1\} >,$$

where $x \xrightarrow{k} y = \begin{cases} y, & \text{if } x \leq y \text{ and } (x,y) \neq 1, \text{ i.e. } x \text{ and } y \text{ are not} \\ & \text{mutually prime numbers, where } x, y \notin \{0, n-1\} \\ x \to y, & \text{otherwise.} \end{cases}$

Thus, $x \xrightarrow{k} y$ differs from $x \to y$ in that $x \xrightarrow{k} y$ does not always take the designated value $n-1$ when $x < y$ and if $x = y$ then $x \xrightarrow{k} y = n-1$ only when $x \in \{0, 1, n-1\}$.

Logic K_n is defined in analogy with Ł$_n$, and $\mathcal{K}_n$ is the set of all matrix functions from K_n.

To prove the theorems the following two properties of divisibility relation (p.d.r.) are required:

I (p.d.r.). If x and y divide by z then their sum $x+y$ divides z.

II (p.d.r.). If x and y divide by z with $x < y$ then their difference $x - y$ also divides z.

Theorem 1. *For any* $n-1 \geq 2$, $n-1$ *is a prime number iff* $n-1 \in \mathcal{K}_n$.

Proof. *Sufficiency*: if $n-1$ is a prime number then $n-1 \in \mathcal{K}_n$:

$$\sim ((x \xrightarrow{k} y) \xrightarrow{k} \sim (x \xrightarrow{k} y)) \xrightarrow{k} (\sim (x \xrightarrow{k} y) \xrightarrow{k} (x \xrightarrow{k} y)) = n-1.$$

Necessity: if $n-1$ is not prime then it has divisors (one at least) different from 1 and $n-1$. Let d^* be one of such divisors of $n-1$ and let D^* be the set of elements $m_i \cdot d^*$ with $m_i = 1, 2, \ldots, n-1/d^* - 1$. We shall show that the set D^* is closed relative to $\sim x$ and $x \xrightarrow{k} y$.

Let $x \in D^*$ and $x = m_i \cdot d^*$. Then $\sim x = \sim (m_i \cdot d^*) = n - 1 - m_i \cdot d^*$. It follows from II(p.d.r.) that $(n - 1 - m_i \cdot d^*) \mid d^*$. Hence $\sim x \in D^*$.

Let $x, y \in D^*$ and $x = m_i d^*$, $y = m_j \cdot d^*$. Then $x \xrightarrow{k} y = m_i \cdot d^* \xrightarrow{k} m_j \cdot d^*$. We have two subcases:

(1) $m_i \leq m_j$. By definition $x \xrightarrow{k} y$, $m_i \cdot d^* \xrightarrow{k} m_j \cdot d^* = m_j \cdot d^*$. Hence $x \xrightarrow{k} y \in D$.

(2) $m_i > m_j$. By definition $x \overset{k}{\to} y$, $m_i \cdot d^* \overset{k}{\to} m_j \cdot d^* = n-1-m_i \cdot d^* + m_j \cdot d^*$. It follows from II (p.d.r.) and I (p.d.r.) that $(n-1-m_i \cdot d^* + m_j \cdot d^*) \mid d^*$. Hence $x \overset{k}{\to} y \in D^*$.

Consequently there is no superposition $f(x)$ of function $\sim x$ and $x \overset{k}{\to} y$ such that $f(x) = n-1$ if $n-1 \neq p$.

Theorem 1 is proved. Thus, a prime number is defined by means of the class of tautologies, i.e. an arbitrary natural number $n-1$ such that $n-1 \geq 2$ is a prime iff the corresponding matrix construction is a logic (in the above sense).

THEOREM 2. *For any $n-1 \geq 2$ such that $n-1$ is a prime number, $\mathcal{K}_n = \mathcal{L}_n$.*

PROOF. Since $\mathcal{K}_n \neq P_n$, in virtue of the result by Bochvar and Finn concerning functional properties of $Ł_n$ (see also [2]) to prove the theorem we have to express $x \to y$ by means of superposition of $\sim x$ and $x \overset{k}{\to} y$. It can be done aa follows:

$x \overset{1}{\to} y = \sim((y \overset{k}{\to} x) \overset{k}{\to} \sim(y \overset{k}{\to} x)) \overset{k}{\to} (x \overset{k}{\to} y)$;
$x \overset{1}{\vee} y = (x \overset{1}{\to} y) \overset{1}{\to} y$;
$x \overset{2}{\to} y = ((x \overset{k}{\to} y) \overset{k}{\to} (\sim y \overset{k}{\to} \sim x)) \overset{1}{\vee} ((\sim y \overset{k}{\to} \sim x) \overset{k}{\to} (x \overset{k}{\to} y))$;
$x \overset{k}{\vee} y = (x \overset{k}{\to} y) \overset{k}{\to} y$;
$x \vee y = (x \overset{k}{\to} y) \overset{1}{\vee} (y \overset{k}{\to} x) = max(x, y)$;
$x \overset{3}{\to} y = (x \overset{k}{\to} y) \vee (\sim y \overset{k}{\to} \sim x)$;
$x \overset{3}{\vee} y = (x \overset{3}{\to} y) \overset{3}{\to} y$;
$x \overset{4}{\to} y = ((x \overset{3}{\vee} y) \overset{2}{\to} (x \vee y)) \overset{1}{\to} (x \overset{3}{\to} y)$;
$x \to y = (x \overset{4}{\to} y) \overset{1}{\vee} (\sim y \overset{4}{\to} \sim x) = min(n-1, n-1-x+y)$.

Theorem 2 is proved. We want to point out that functional equivalence of sets $\mathcal{K}_n$ and $\mathcal{L}_n$ is proved only for the case when $n-1$ is a prime number, i.e. for a sequence of prime numbers rather than for the whole natural series. Hence the complexity of analytical expression which proves this equivalence and which contains 21345281 occurrences of implication $\overset{k}{\to}$ and, thus, the complexity of distributing prime numbers in natural series is reflected.

References

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