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T-NORMS AND φ-OPERATORS AS TRUTH FUNCTIONS OF MANY VALUED CONNECTIVES

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The choice of connectives for many valued propositional logics suitable for theoretical and applicational interests in most cases is an open problem up to now. We will not offer a general solution here, but support the point of view of some recent developments in fuzzy set theory that the triangular norms – t-norms for short – of Schweizer/Sklar [3] and the φ-operators of Pedrycz [2] represent quite general classes of connectives at least for many valued logics with truth value set the real interval [0,1] (or also some suitable finite subset). Special classes of t-norms and general properties of them have been discussed in some papers just recently, e.g. by Dombi [1], Weber [4], Yager [5] to mention only a few of them.

Definition 1. (i) A t-norm $\underline{t}$ is a binary operation in [0,1] which is commutative and associative, and for which it holds always true that

$$x\underline{t}1 = x, \qquad\qquad x\underline{t}0 = 0,$$

$$x \leq y \Rightarrow x\underline{t}z \leq y\underline{t}z$$

(ii) A φ-operator φ connected with a theorem $\underline{t}$ is a binary operation in [0,1] such that it is always true that

$$y \leq z \Rightarrow x\varphi y \leq x\varphi z,$$

$$x\underline{t}(x\varphi y) \leq y \leq x\varphi(y\underline{t}y).$$

Here we have used $\Rightarrow$ for two-valued implication of our metalanguage; later on we use $\Leftrightarrow$ in analogous for equivalence.

As a crucial lemma one proves from these definitions that

$$x\varphi y = sup\{z : x\underline{t}z \le y\},$$

such that with each t-norm only one φ-operator is connected which also is antimonotonous in the first argument. Furthermore

$$x \le y \Rightarrow x\varphi y = 1.$$

In case of a lower semicontinous t-norm $\underline{t}$ – $LSC(\underline{t})$ for short – the corresponding φ-operator φ has property:

$$LSC(\underline{t}) \Rightarrow (x \le y \Leftrightarrow x\varphi y = 1).$$

In the following, to simplify notation, φ always will be the φ-operator connected with the t-norm considered.

The definition, these, and tho subsequent properties make it obvious that t-norma are generalized conjunctions and φ-operators generalized implication operators. Hence, all the inequalities considered later on may be viewed as stating that Borne implicational statement has truth value 1, which value in fuzzy set theory (implicitly) is supposed to be the only designated one.

PROPOSITION 1. *The following formulae hold true:*

(i) $x\underline{t}(x\varphi y) \le y$,
(ii) $x\varphi(y\varphi(x\underline{t}y)) = 1$,
(iii) $(x\underline{t}y)\varphi z \le x\varphi(y\varphi z)$,
(iv) $x\varphi y \le (x\underline{t}z)\varphi(y\underline{t}z)$.

PROPOSITION 2. *Suppose* $LSC(\underline{t})$. *Then there hold true*

(i) $(x\underline{t}y)\varphi z = x\varphi(y\varphi z)$,
(ii) $(x\varphi y)\underline{t}(y\varphi z) \le x\varphi z$.
(iii) $(x\varphi y)\underline{t}(u\varphi v) \le (x\underline{t}u)\varphi(y\underline{t}v)$.

For some more results we introduce a negation operation and the usual quantifiers. Yet, it should be noted that from fuzzy set theory presently we can get hints only scarcely regarding the choice of such a negation operation.

Definition 2. By $\bigwedge$ and $\bigvee$ we denote resp. the infimum and supremum operation (which correspond to many valued generalization and existential quantification) and we put

$$-x =_{df} x\varphi 0.$$

Proposition 3. *Suppose that y is independent of ξ but x may depend of ξ. Then there hold true*

(i) $\bigwedge_\xi (y\varphi x) = y\varphi(\bigwedge_\chi x)$,
(ii) $(\bigvee_\chi x)\varphi y \leq \bigwedge_\chi (x\varphi y)$,
(iii) $\bigvee_\chi (x\varphi y) = (\bigwedge_\chi x)\varphi y$.
(iv) $\bigvee_\chi (y\varphi x) = y\varphi(\bigvee_\chi x)$,
(v) $(\bigwedge_\chi x_1)\underline{t}(\bigwedge_\chi x_2) \leq \bigwedge_\chi (x_1 \underline{t} x_2)$,
(vi) $\bigvee_\chi (x_1 \underline{t} x_2) \leq (\bigvee_\chi x_1)\underline{t}(\bigvee_\chi x_2)$.

Proposition 4. *Suppose $LSC(\underline{t})$ and furthermore the same as in Proposition 3. Then*

(i) $\bigwedge_\chi (x\varphi y) = (\bigvee_\chi x)\varphi y$,
(ii) $\bigvee_\chi (x\underline{t}y) = (\bigvee_\chi x)\underline{t}y$.

To get a corresponding property to Proposition 4(ii) also for $\bigwedge$, we have to suppose instead of $LSC(\underline{t})$ that $\underline{t}$ is upper semicontinuous.

Finally, with respect to negation e.g. some versions of contraposition hold true besides the simple properties

$$x \leq --x, \qquad -x = ---x,$$

$$x \leq y \Rightarrow -y \leq -x.$$

PROPOSITION 5. *Suppose $LSC(\underline{t})$. Then*

(i) $x\varphi y \leq (-y)\varphi(-x)$,
(ii) $x\varphi(-y) = y\varphi(-x)$,
(iii) $x\underline{t}(-x) = 0$,
(iv) $\bigvee_{\chi}(-x) \leq -\bigwedge_{\chi} x$,
(v) $-\bigvee_{\chi} x \leq \bigwedge_{\chi}(-x)$.

As consequences of these last two inequalities we get the relations

$$\bigvee_{\chi} - - x \leq - \bigwedge_{\chi} -x \leq - - \bigvee_{\chi} x.$$

$$- - \bigwedge_{\chi} x \leq - \bigvee_{\chi} -x \leq \bigwedge_{\chi} - - x.$$

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