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# AN INTERPRETATION OF A CERTAIN FRAGMENT OF ARITHMETIC IN SOME PROPOSITIONAL CALCULUS

Here it is shown that a certain non-trivial fragment of arithmetic can be interpreted in some propositional calculus. Arithmetical formulae are interpreted there as sets of rules of our calculus. Since among interpretable arithmetical formulae there are some which are as yet unsolved arithmetical problems, we are unable to say about some sets of rules whether they contain, any rules of our calculus or not. I think that there is'a conviction about sufficiency of arithmetical tools for solving problems of propositional calculi. In view of the above remark this conviction is false. Even so we can show some interesting properties of our calculus. Eventually we show that our calculus determines some topological space, so that we can speak about an interpretation of the mentioned fragment of arithmetic in this topology.

Definitions of notions, used here but not defined, can be found in [1]. The only difference is in this that we will use the notion of relation of consequence instead of the notion of operation of consequence. We will say consequence instead of relation of consequence.
If consequences $\vdash_1$, $\vdash_2$, ... are determined by finite matrices and $\vdash$ is a consequence such that $\vdash = \bigcap\{\vdash_1, \vdash_2, \ldots\}$, then we say that a consequence - is strongly finite approximable.

Let $Ln = (Fr, \oplus)$ be an algebra" of formulas where $\oplus$ is a two argument connective and $Fr$ the set of all formulas formed by means of $\oplus$ and prepositional variables. Let $^-$ be a function of length of formulas from $F$r defined as follows; for all formulas $a, b$

$\bar{a} = 1$ iff $a$ is a propositional variable
$\overline{a \oplus b} = \bar{a} + \bar{b}$

By the same symbol $^{-}$ we denote an operation on sets of formulas defined as follows: for every $X \subseteq Fr$

$$\bar{X} = \begin{cases} min\{a : a \in X\} & \text{if } X \neq \emptyset \\ \infty & \text{otherwise} \end{cases}$$

Let $|\!\vdash$ be a relation between sets of formulas and formulas defined as follows: for all $X \subseteq Fr$, $a \in Fr$

$X \mid\!\vdash a$ iff $\bar{X} \leq \bar{a}$

Accept the following abbreviations '$a \odot b$' instead of '$\underbrace{b \oplus \ldots \oplus b}_{a}$' and '$\underline{n}$' ($n = 1, 2, ...$) instead of '$\underbrace{u \oplus (u \oplus \ldots \oplus (u \oplus u) \ldots)}_{n}$' where $u$ is any fixed propositional variable of $Ln$.
Let for all $a, b, c \in Fr, a \oslash b = a$ iff $\bar{b} = 1$ and $a \oslash (b \oplus c) = (\ \oslash b) \odot (a \oslash c)$[1].
Speaking about arithmetic we make the following two assumptions: firstly, it is constructive and secondly, its expressions do not contain zero and variables range over natural numbers $1, 2, \ldots$. So, a formula $(\forall x)A(x)$, where $x$ is the only variable of formula $A(x)$, is a thesis of this arithmetic iff formulas $A(1), A(2), ...$ are its theses. This system will be denoted by the symbol $Ar$. Speaking about natural numbers, whose set is denoted by the letter $N$, we mean their set without zero. If $B$ is an arithmetical formula then $B$ denotes the formula $B$ preceded by universal quantifiers binding all free variables of $B$.
Let $Trm$ be a set of terms formed in the usual manner by means of:

natural numbers from $N$,
variables ranging over elements of $N$,
operations: $+$ (addition), $\cdot$ (multiplication) and $|$ (involution)[2].

For simplicity of further considerations we assume that the sets of propositional variables of $Ln$ and variables ranging over natural numbers are of the same cardinality. Also for simplicity we will denote these two kinds of variables by the same letters $x, y, v, z$. (Obviously it will not lead to

[1] We will often write $a^b$ instead of $a \oslash b$.

[2] $|$ is defined analogously as $\oslash$ (see above). Often instead of $\alpha \mid \beta$ we will write $\alpha^\beta$ ($\alpha, \beta \in Trm$).

misunderstanding, because of the context in which they will be used).
If $\alpha \in Trm$ then by the symbol $\alpha^*$ we will denote a formula $Fr$ obtained from $\alpha$ by the respective replacement:

of symbols $1, 2, ...$ by symbols $\underline{1}, \underline{2}, ...$
of symbols $+, \cdot, |$ by symbols $\oplus, \odot, \oslash$.

By the letter $F$ we denote the set of all substitutions in arithmetical formula of the form $\alpha \leq \beta$ or $\alpha = \beta$ $(\alpha, \beta \in Trm)$ such that natural numbers replace variables (i.e. $f \subseteq F$ if $fTrm \subseteq N$ and $f$ preserves operations $+, \cdot, |$ and relations $\leq, =$).

$$R_f(\alpha \leq \beta) = \{(\{f\alpha)^*\}, (f\beta)^*\}$$
$$R_f(\alpha = \beta) = R_f(\alpha \leq \beta) \cup R_f(\beta \leq \alpha) \qquad (\alpha, \beta \in Trm,\ f \in F)$$

$R(\alpha \leq \beta) = \bigcup\{R_f(\alpha \leq \beta) : f \in F\}$ and $R(\alpha = \beta) = R(\alpha \leq \beta) \cup R(\beta \leq \alpha)$.

It can easily be proved that the relation $|\!\vdash$ is a consequence.

THEOREM 1. *A formula* $| \ \alpha \leq \beta \ |$ *is an* $Ar$*-thesis iff* $R(\alpha \leq \beta)$ *is a* $|\!\vdash$ *- rule, and a formula* $| \ \alpha = \beta \ |$ *is an* $Ar$*-thesis iff* $R(\alpha = \beta)$ *is a* $|\!\vdash$ *-rule* $(\alpha, \beta \in Trm)$.

As can be seen, the above theorem indicates a connection between a fragment of $Ar$-arithmetic and the propositional calculus determined by the consequence $|\!\vdash$. Sometimes it is easy to decide whether a rule which is an interpretation of some arithmetical formula is or is not a $|\!\vdash$ -rule. For example it is obvious that of the following rules:

$R(2xy \leq x^2 + y^2)$, $R((x + y)z = xz = yz)$, $R(x^{v+2} + y^{v+2} = z^{v+2})$, $R(x^3 + y^3 = z^3 + 30)$ – the first two are and the rest are not $|\!\vdash$ -rules. Yet sets of rules can be indicated which are not sets of $|\!\vdash$ -rules but it is impossible, at least today, to decide whether some rule (from this kind of set) is a $|\!\vdash$ -rule. To demonstrate this we can consider the following set of rules: $\{R_f(x^{v+2} + y^{v+2} = z^{v+2}) : f \in F\}$. The sum of this set gives the rule $R(x^{v+2} + y^{v+2} = z^{v+2})$ which, as we have mentioned above, is not a $|\!\vdash$ -rule. By the proof of Theorem 1 we can observe that the question – whether some rule from the set $\{R_f(x^{v+2} + y^{v+2} = z^{v+2}) : f \in F\}$ is a $|\!\vdash$ -rule – is equivalent to the question – whether some formula from the set

$\{f(x^{v+2} + y^{v+2} = z^{v+2}) : f \in F\}$ is an $Ar$-thesis. This last question, by the definition of $F$ and the system $Ar$, can be expressed as follows:

(1) Are there $v \geq 3, x, y, z$ such that $x^v + y^v = z^v$?

But this question is a question about the truth of Fermat's Great Theorem.
Similarly the question – whether some rule of the set $\{R_f(x^3+y^3 = z^3+30) : f \in F\}$ is a $|\!\vdash$ -rule – is equivalent to the following question:

(2) Are there $x, y, z$ such that $x^3 + y^3 = z^3 + 30$?

A mathematician will say that question (1) and (2) are non-trivial arithmetical problems, Then we know the answer to these questions then we will know whether the respective sets of rules contain any $|\!\vdash$ -rules.
Let us note, however, that it can be stated a priori that there are many non-trivial problems of the following type:

Are there $x_1, \ldots, x_n$ such that $\alpha = \beta$?

(where $x_1, ..., x_n$ are all the free variables of the formula $\alpha = \beta$). It follows from this, I think, that we shall 'never' be able to say about every set of rules of the type $\{R_f(\alpha = \beta) : f \in F\}$ that some of its rules is a $|\!\vdash$ -rule.

For every consequence there exists the greatest structural consequence $\subseteq$ in the first (cf. [1]). Let for $|\!\vdash, \models$ be such a consequence, i.e. $\models \subseteq |\!\vdash$ and $\models$ is the greatest structural consequence having this property.

Theorem 2. *$\models$ is a strongly finitely approximable consequence.*

Remark. It can be shown that if the consequence $\models$ determined by finite set of standard rules, then the calculus $(Ln, \models)$ is decidable; and it can be shown that the problem of decidability for this calculus is connected with considerations of rules of the types: $R_f(...), R(...)$ having interpretations in the system $Ar$.

So, we can state non-trivial the following:

Problem. Is there $\models$ a consequence determined by finite set of standard rules?

Let, now, $\underline{C} : 2^{Fr} \to 2^{Fr}$ be an operation defined as follows $\underline{C}(X) = \{a : X \,|\!\vdash a\}$ for every $X \subseteq Fr$). It can easily be proved that $\underline{C}$ is consequence operation, i.e. for all $X, Y \subseteq Fr$, $X \subseteq \underline{C}\underline{C}(X) \subseteq \underline{C}(X \cup Y)$. Operation $\underline{C}$ becomes a topological closure if additionally the following conditions are satisfied: $\underline{C}(\emptyset) = \emptyset$ and $\underline{C}(X \cup Y) \subseteq \underline{C}(X) \cup \underline{C}(Y)$. So the pair $(Ln, \underline{C})$ is a topological space in which can be interpreted analogously the mentioned fragment of arithmetic $Ar$, i.e. all our considerations for the calculus $(Ln, \Vdash)$ hold for the space $(Ln, \underline{C})$ where instead of $\Vdash$ we use $\underline{C}$

## References

[1] R. Wójcicki, *Matrix approach in methodology of sentential calculi*, **Studia Logica**, Vol. XXXII, (1973), pp. 7–37.

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