The following logics are the most noteworthy from the perspective of the calculus of combinators: the Hilbert’s positive implicational logic (i.e. purely implicational fragment of the intuitionistic propositional calculus), the Church’s weak theory of implication (i.e. purely implicational fragment of the relevant system $R$), the $BCK$-logic, and the $BCI$-logic. Their significance is due to a certain correspondence between combinators and implicational formulas (see for example [1]). The first three logics mentioned have been immensely investigated but it was not so in case of the remaining one. The $BCI$-logics was mentioned by A. N. Prior in the second edition of his Formal Logic of 1962 where it was credited to C. A. Meredith and dated in 1956 (see [4]). According to the definition the $BCI$-logic is determined by the following rules:

\((B) \vdash (\alpha \rightarrow \beta) \rightarrow ((\gamma \rightarrow \alpha) \rightarrow (\gamma \rightarrow \beta))\),
\((C) \vdash (\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow (\beta \rightarrow (\alpha \rightarrow \gamma))\),
\((I) \vdash \alpha \rightarrow \alpha\),

and $\alpha, \alpha \rightarrow \beta \vdash \beta$.

The set of theorems of the $BCI$-logic is decidable as it was shown by S. Jaśkowski in 1963 (see [3]).
2.

In 1966 K. Iseki introduced the concept of $BCI$-algebra as an algebraic counterpart of the $BCI$-logic. In his paper [2] in the introduction Iseki writes: “In this note, we shall consider a new algebra induced by the $BCI$-system of propositional calculus by C. A. Meredith quoted into A. N. Prior, Formal Logic ([4], p. 316)”. Let us cite the Iseki’s definition: “Let $M = (X, 0, \ast)$ be an abstract algebra consisting of a set $X$ with an element $0$ and binary operation $\ast$. If $M$ satisfies the following conditions $BCI$ 1-5, it is called a $BCI$-algebra:

$BCI 1 \ (x \ast y) \ast (x \ast x) \leq x \ast y,$
$BCI 2 \ x \ast (x \ast y) \leq y,$
$BCI 3 \ x \leq x,$
$BCI 4 \ x \leq y, y \leq x$ imply $x = y,$
$BCI 5 \ x \leq 0$ implies $x = 0,$

where $x \leq y$ means $x \ast y = 0$.

3.

Unfortunately the above definition fails to describe a class of algebras adequate for $BCI$-logic or even for the set of theorems of $BCI$-logic, and thus the term $BCI$-algebra is misleading. One can prove that $BCI$-algebras as defined in [2] determine a logic which is stronger then $BCI$-logic. To be precise we have the following.

Completeness Theorem. The class of $BCI$-algebras is strongly adequate for the logic determined by the following rules:

$\vdash (\alpha \rightarrow \beta) \rightarrow ((\gamma \rightarrow \alpha) \rightarrow (\gamma \rightarrow \beta)),$
$\vdash (\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\beta \rightarrow (\alpha \rightarrow \gamma)),$
$\vdash \alpha,$
$\alpha, \alpha \rightarrow \beta \vdash \beta,$
$\alpha, \beta \vdash \alpha \rightarrow \beta.$
4.

It should be mentioned that conditions used by Iseki in his definition of \(BCI\)-algebras are not independent. Equivalently one can define \(BCI\)-algebras as all algebras \(K = (A, 0, \ast)\) of type \((0, 2)\) satisfying:

\[
\begin{align*}
\ast1 & \quad a \ast 0 = a, \\
\ast2 & \quad ((a \ast b) \ast (a \ast c)) \ast (c \ast b) = 0, \\
\ast3 & \quad a \ast b = b \ast a = 0 \text{ iff } a = b.
\end{align*}
\]

For the proof it suffices to note that \((a \ast (a \ast b)) \ast b = ((a \ast 0) \ast (a \ast b)) \ast (b \ast 0) = 0.\)

5.

The above definition makes it clear that \(BCI\)-algebras form a quasivariety while the result of A. Wroński [5] yields the following

**FACT.** \(BCI\)-algebras do not form a variety.

References