Janusz Czelakowski

ALGEBRAIC ASPECTS OF DEDUCTION THEOREMS

By a *sentential logic* we understand a pair

\[(S, C)\],

where \(S\) is a sentential language, i.e. an absolutely free algebra freely generated by an infinite set \(p, q, r, \ldots\) of sentential variables and endowed with countably many finitary connectives \(\&_1, \&_2, \ldots\) and \(C\) is a consequence operation on \(S\), the underlying set of \(S\) (= the set of formulas of \(S\)), satisfying the condition of structurality: \(eC(X) \subseteq C(eX)\), for every endomorphism \(e\) of \(S\) and for every \(X \subseteq S\). If no confusion is likely we shall identify a logic \((S, C)\) with its consequence operation \(C\). A logic \(C\) is *standard* if

\[C(X) = \bigcup \{C(Y) : Y \subseteq X \& Y \text{ is finite} \},\]

for all \(X \subseteq S\).

Let \(S\) be a sentential language and let \(P\) be a set of formulas of \(S\) in two distinct variables \(p\) and \(q\). Let us denote by \(P(\alpha, \beta)\) the set of formulas which result by the simultaneous substitution of \(\alpha\) for \(p\) and \(\beta\) for \(q\) in all formulation in \(P\).

Let \(C\) be a logic in \(S\). We shall say that \(P\) is a *uniform deduction theorem scheme for \(C\)* if for every set \(X \subseteq S\) and every formulas \(\alpha, \beta \in S\)

\[\beta \in C(X, \alpha) \text{ iff } P(\alpha, \beta) \subseteq C(X).\]

If \(P\) is a uniform deduction theorem scheme for \(C\), then we shall also say that \(C\) admits the deduction theorem \((C\) admits DT, for short) with the scheme \(P\).
Theorem 1. Let $C$ be a standard logic in a language $S$ and let $P$ be set of formulas of $S$ in two variables. The following conditions are equivalent:

(a) $C$ admits DT with the scheme $P$;
(b) $C$ satisfies the following clauses:

(i) $P(p, p) \subseteq C(\emptyset)$
(ii) $P(q, p) \subseteq C(p)$
(iii) $q \in C(\{p\} \cup P(p, q))$
(iv) for every finite nonempty set of formulas $\gamma_1, \ldots, \gamma_n$ of $S$ and for every $\alpha \in S$, if $\alpha \in C(\{\gamma_1, \ldots, \gamma_n\})$, then $P(p, \alpha) \subseteq C(P(p, \gamma_1) \ldots P(p, \gamma_n))$.

Theorem 2. Let $C$ be a standard logic in $S$ and let $P$ be a (possibly infinite) uniform deduction theorem scheme for $C$. Then there exists a finite subset $P_0$ of $P$ such that $P_0$ is also a uniform deduction theorem scheme for $C$.

Given a logic $(S, C)$ and an algebra $A$ similar to $S$, we say that a subset $\nabla$ of $A$ is a deductive filter on $A$ (relative $C$) if the matrix $(A, \nabla)$ validates the logic $C$, i.e., if for every $X \subseteq S$ and every $\alpha \in S$, if $\alpha \in C(X)$, then $v\alpha \in \nabla$ whenever $vX \subseteq \nabla$, for every homomorphism $v : S \rightarrow A$.

The set $F_C(A)$ of all deductive filters on $A$ is a complete lattice under the set-theoretic inclusion and $A$ is a greatest member of $F_C(A)$. One easily verifies that if $C$ is the classical logic and $A$ is a Boolean algebra then $F_C(A)$ is the family of ‘ordinary’ filters of $A$.

A deductive filter $\nabla \in F_C(A)$ is called compact (or finitely generated) if there is a finite set $X$ such that $\nabla$ is the smallest deductive filter in $F_C(A)$ containing $X$. The family of compact deductive filters on $A$, denoted by $F^\text{comp}_C(A)$, is closed under the lattice-theoretic join in $F_C(A)$; in other words, $F^\text{comp}_C(A)$ is a join-semilattice. If the logic $C$ is standard, then for every algebras $A$ the lattice $F_C(A)$ is algebraic, that is, every deductive filter on $A$ (relative $C$) is a set-theoretic join of compact filters from $F^\text{comp}_C(A)$.

Let $I$ be a join-semilattice. $I$ is called dually-Brouwerian (see [3]) iff for each $a, b \in T$, the dual relative pseudo-complement $a \ast b$ exists, i.e., for all $x \in T$: 
\[ b \leq a \lor x \iff a \ast b \leq x. \]

**Theorem 3.** Let \( C \) be a standard logic in \( S \) admitting DT. Then for every algebra \( A \) similar to \( S \), the join-semilattice \( F_C^{\text{comp}}(A) \) of compact deductive filters on \( A \) is dually Brouwerian.

A logic \( C \) in a language \( S \) is called **filter distributive** (see also [2]) if for every algebra \( A \) similar to \( S \), the lattice \( F_C(A) \) is distributive. Theorem 3 has an interesting corollary.

**Corollary 4.** Let \( C \) be a standard logic admitting DT. Then \( C \) is filter distributive.

The converse of Theorem 3 holds in the case of equivalential logics (see [1]):

**Theorem 5.** Let \( C \) be a standard equivalential logic in a language \( S \). The following assertions are equivalent:

(i) \( C \) admits DT

(ii) For every algebra \( A \) similar to \( S \), the semilattice \( F_C^{\text{comp}}(A) \) is dually Brouwerian.

Let \( N \) be a normal modal system. \( N \) defines two modal logics, i.e., structural **consequence operations** in the modal language, denoted respectively by \( \overline{N} \) and \( \overline{N} \Box \). \( \overline{N} \) is determined by Modus Ponens as the sole primitive non-axiomatic rule of inference and by \( N \) as the set of axioms while \( \overline{N} \Box \) is a strengthening of \( \overline{N} \) obtained by adding the rule of necessitation to the set of rules of \( \overline{N} \).

Each logic \( \overline{N} \) has a uniform deduction theorem scheme. The scheme is provided by the material implication connective.

Given a variable \( p \) define \( \Box^0 p = p \) and \( \Box^{n+1} p = \Box(\Box^n p) \). Let

\[ P^n = \{ \Box^0 p \rightarrow q, \ldots, \Box^n p \rightarrow q \}. \]

Let \( K \) be the Kripke’s modal system. One can show that for every set \( X \cup \{ \alpha, \beta \} \) of modal formulas

\[ \beta \in \overline{K} \Box(X, \alpha) \iff \text{there is a natural } n \text{ such that } P^n(\alpha, \beta) \subseteq \overline{K} \Box(X). \]

It follows from Theorem 5 that \( K \) does not possess an uniform deduction theorem scheme. The same remark applies to the logic \( T \), where \( T \) is the Fey’s modal system.
References


*The Section of Logic*
*Institute of Philosophy and Sociology*
*Polish Academy of Sciences*