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PARACONSISTENCY, PARACOMPLETENESS AND INTENTIONAL CONTRADICTIONS

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In this paper intentional contradictions are dealt with, i.e. antinomies which are arrived at on purpose, and not by chance or mistake. I take for granted that there are antinomic propositions (e.g. the Liar or some other paradoxes) which are characterized as follows:

(1) their truth is possible alongside with the truth of their negations,
(2) from such a statement its negation is inferred.

For instance, the Liar’s self-referential proposition to the effect that it is itself false, as a matter of fact is self-negatory. The former requirement (1) is a semantic one. It can be made compatible with the consistency requirement by either modifying the concept of truth (resp. the predicate: ‘True...’) towards a less rigorous one (cf. G. H. von Wright’s logic of truth TL) or modifying the notion of negation to obtain a negation operator which disobeys the law of contradiction. The present author has been working along these lines in cooperation with J. Sánchez.

As regards the syntactic requirement (2), it is incompatible with mere consistency, i.e. with consistency under the transformation of a formula into its negation (I shall further call this relative inconsistency), but does not necessarily involve absolute inconsistency... The notion* of paraconsistency comes to the rescue. It means relative inconsistency which, however, is by no means generalized to render an arbitrary formula provable and hence the *system as a whole does not explode into overcompleteness or triviality. The vigorous investigations of such inconsistent but non-trivial formal systems is nowadays associated with the names of N. C. A. da Costa, A. Arruda, R. Routley at all but it was L. Wittgenstein who predicted as...
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early as 1930 “a time when there will be mathematical investigations of calculi, containing contradictions, and people will actually be proud of having emancipated themselves from consistency.” PR, p. 332. As a matter of fact dialectical logicians, have always been emancipated from consistency, which does not imply that they have been devoted to triviality. But the problem is how to reconcile of mathematical investigations and formal systems to the method and achievements of dialectics.

My approach to the problem of how to prevent a formal system, from being trivialized by a contradiction provable in it, involves the notions of paraconsistency and paracompleteness. The latter is related to the former in the following sense, or way. I shall call a system $S$, paracomplete, if there is a formula $\beta$ which is undecidable in $S$ and the adding of $\beta$ to the axioms of $S$ makes this system paraconsistent. To put it in another way, $\beta$ is neither provable, nor refutable in $S$, but the adding of $\beta$ to the axioms of $S$ renders both $\beta$ and its negation $-\beta$ provable in this extension of $S$. In the process of extending a certain formal system by adding new axioms to it we do not necessarily have to include the newly added axioms in the domain of all rules of inference, under which the initial system $S$, is closed. So the extension itself is not necessarily closed under all the rules of the system extended. It seems quite enough for it to be closed under one or two of them – they are therefore considered main rules of the new system $S'$. As for the rest, they are restricted rules, their domains being proper subclasses of the class of all theorems in the system. We may also introduce some new restricted rules of inference which include in their domain the newly added axioms but exclude some axioms of the initial system $S$. This conforms the process of developing theoretical knowledge by adding hypotheses and accounts of observations which are not necessarily included in the domains of older theories. For example, relativistic mechanics is not reduced to classical mechanics, no matter that the case when $v/c \to 0$ can be regarded as a limiting case sui generis.

On the other hand, dependence characteristic are juxtaposed to the theorems (following E. Vojshvillo’s approach to natural deduction as well as to the premises in the formulation of the very rules of inference. These characteristics, written in quotation marks, to the right of the corresponding formula, indicate the axiom from which the theorem in question has been deduced. A rule of inference cannot use as premises the theorems, derived from axioms which are not included in the list of its dependence characteristics.
Some antinomic propositions are content-determined ones (they might as well be elementary or atomic propositions.) They should be represented by single propositional variables, as far as the syntax of a propositional calculus is concerned. To put it in another way, such statements should be regarded as synthetic, rather than analytic ones. So I am going to differentiate between analytical truths (making use of axiom-schemata to express them in their universal validity, i.e. holding good for an arbitrary proposition, resp. any formula of the system), and on the other hand — a kind of synthetic truth of an elementary statement. I shall therefore postulate, in addition to the axiom-schemata which stand for analytical (logical) truths, a single propositional variable to represent the content-determined, non-analytical (someone may refer to it as non-logical) truth. I shall try to organize the formula system in such a way that this unusual axiom (non axiom-scheme) shall not trivialize it, although the negation of this axiom is also a theorem of the system in question.

The system $HPS_4$, of which I am going to present but an outline, can illustrate this approach. Its primitive symbols include an infinite set of propositional variables: $p,q,r,\ldots,p_1,q_1,r_1,\ldots$; technical symbols: $(,)$ used as usual, with normal conventions for bracketing and omitting brackets; the following operators: $\&$, $\lor$, $\neg$, $M$, i.e. conjunction, disjunction, negation and a modal operator for possibility.

The usual formation rules hold (resp. the usual recursive definition of the notion of a formula). Abbreviations are introduced for material implication and the corresponding equivalence. The transformation rules include the axiom-schemata of the well known modal system $S_4$ with the addition of one more axiom — a single propositional variable $p$, as well as rules of inference, relativized with respect to the dependence y characteristic of their premises:

\begin{align*}
\text{ax.1} & \quad \text{The tautologies of classical propositional calculus,} \\
\text{ax.2} & \quad A \rightarrow MA, \\
\text{ax.3} & \quad MA \lor MB, \\
\text{ax.4} & \quad MMA \rightarrow MA, \\
\text{ax.5} & \quad p,
\end{align*}

where $A,B$ are meta-variables, ranging over arbitrary formulas, while $p$ is a propositional variable. The rules of inference are:

R1. $\quad A^*ax.1 - ax.4^*, (A \rightarrow B)^*ax.1 - ax.4^* \vdash B^*ax.1 - ax.4^*$
R2. \( A^{\text{ax.}2 − \text{ax.}5}, (A \rightarrow B)^{\text{ax.}2 − \text{ax.}5} \vdash B^{\text{ax.}2 − \text{ax.}5} \)

R3. If \( p' \) is a propositional variable in a formula \( A \), then from \( A \) infer the substitution of \( -p' \) for every token of \( p' \) in \( A \), with the same dependence characteristics as \( A \) itself.

R4. \( A^{\text{ax.}1 − \text{ax.}5} \vdash NA^{\text{ax.}1 − \text{ax.}5} \)

R5. \( A^{\text{ax.}1 − \text{ax.}5}, B^{\text{ax.}1 − \text{ax.}5} \vdash (A \& B)^{\text{ax.}1 − \text{ax.}5} \)

where \( \rightarrow, \leftrightarrow \) are the symbols for material implication and equivalence respectively and \( N \) is a modal operator, dual with respect to \( M \), i.e. \( NA =df −M − A \). The notions of proof and provable formula are as usual, having in mind the specific status of the rules of inference: they can only use as premise theorems which have been deduced from the axiom, listed in their dependence characteristics, e.g. \( \text{ax.}1 − \text{ax.}4 \), i.e. \( \text{ax.}1, \text{ax.}2, \text{ax.}3, \text{ax.}4 \) but not \( \text{ax.}5 \).

I expect this system to be a paraconsistent one, but the burden of proof lies with me. G. Priest has suggested a very simple way of approaching such a proof: to interpret \( HPS4 \) into one of the paraconsistent modal logics available and, as regards semantics-to give the propositional variable in \( \text{ax.}5 \) an evaluation “true-and-false”.

There are good reasons to expect that the contradictions provable in \( HPS4 \) will not spread to trivialize the system as a whole. The point is that these contradictions, namely \( p \) and \( −p \) both being theorems by way of \( \text{ax.}5 \) and \( R3. \), and consequently the provability of \( (p \& −p) \) by way of \( R5. \), have been derived from \( \text{ax.}5 \) which is not included in the dependence characteristics of \( R1 \). It is in the domain of this rule that the theorem \( ((p \& −p) \rightarrow B) \) is to be found, its dependence characteristics being “\( \text{ax.}1 \)”, i.e. classical propositional logic. This, in turn, prevents the above mentioned theorem (known as D. Scotus law) from being a premise of \( R2 \). Thus neither of the two detachment rules has both premises needed for the inference of an arbitrary formula \( B \) from the contradiction \( (p \& −p) \): \( R1 \) cannot use as a premise D. Scotus law, although it is a theorem of the same system. So are all the theorem of classical propositional logic and of modal logic \( S4 \) (instead of classical tautologies we might as well use the intuitionistic ones, or vary the modal axioms to obtain different modal logics). The most interesting thing about the approach proposed is that the resulting system incorporates some well known formal system and for all that it is tolerant to some contradictions (at least in the sense of not spreading them by means of modus ponens and substitution).
With the help of ax.5 some interesting theorems are proved in addition to the above mentioned contradiction. For example, from ax.2 and ax.5, by way of R2, \( Mp \) is derived, and from it, by way of R3, we obtain \( M - p \). In view of R5 their conjunction is proved: \((Mp \& M - p)\). So the proposition \( p \) turns out to be contingent. But on the other hand it is also necessary, in view of ax.5 and R4. Furthermore, its contingency is necessary, in view of R4: \( N(Mp \& M - p) \). The formula \((\neg p \& Mp)\) is also a theorem and it reads that the truth of \( p \) is possible alongside with the truth of its negation. (The same holds for \( \neg p \)). The formulae \((Np \& M - p)\) and \((N - p \& Mp)\) are also provable contradictions, bearing in mind that the operators \( N \) and \( M \) are dual to each other. The modal explication of these contradictions may contribute to the understanding of the process of motion.

Let us read the modal operators \( N \) and \( M \) as ‘already’ (yet) and ‘still’ respectively. Let, further, in the context of Zeno’s fifth paradox (the one referred to by Diogenes Laertius and Sextus Empiricus) \( p \) be a proposition to the effect that the moving object is occupying a certain place at a given moment of time. The dialectical solution of this paradox is that it is both occupying and not occupying the place in question at that very moment of time – it is plausible, no matter how absurd it may sound to ‘common sense’. The explication of the two contradictions: \((Np \& M - p)\) and \((N - p \& Mp)\) seems to contribute to the feasibility of the dialectical model of motion. The former reads that the moving object is already occupying and still not occupying the place in question at the given moment of time, i.e. it is beginning to occupy it. The latter reads that the moving object is already not occupying and still occupying it, i.e. ceasing to occupy it. Both beginning and ceasing are processes of becoming, in terms of dialectical logic. So if a moving object is occupying a given place and at the same time keeps moving, it must have necessarily passed through a state of beginning to occupy the place in question and on the other hand – of ceasing to occupy it – otherwise it is not moving through this place but rather rests there. Motion is itself a process of becoming: a contradiction is constantly emerging and being resolved in its capacity of a discrete state of affairs.

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