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ON TWO PROPERTIES OF STRUCTURALLY COMPLETE LOGICS

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Preliminary notions

Let $S = \langle S, f_1, \ldots, f_n \rangle$ sentential language $C$, possibly with index, denotes a standard consequence. $Sb$ is the consequence operation determined by the rule of substitution. By $L(C)$ we denote the set of all Lindenbaum’s extensions of the consequence $C$ ($L(C) = \{ X \subseteq S : C(X) \neq S \text{ and } C(X \cup \{\alpha\}) = S \text{ for every } \alpha \notin X \}$). $End(S)$ is the set of all endomorphisms of $S$. Let $\mathcal{U}$ be the consequence operation induced by a matrix $\mathcal{U}$. The symbol $E(\mathcal{U})$ stands for the set of all formulas which are valid in $\mathcal{U}$. We also write $\mathcal{U} \subseteq \mathcal{U}$, iff $\mathcal{U}$ is a submatrix of $\mathcal{U}_1$. In this paper we assume that in every functionally complete matrix $\mathcal{U}(A, D)$ the set $D$ is proper non-empty subset of the domain $A$ of $A$. By a rule of inference we mean a non-empty subset of $2^S \times S$. A rule $r$ is finitary iff for every $X \subseteq S$ and every $\alpha \in S$ : if $\langle X, \alpha \rangle \in r$, then $X$ is finite. A rule $r$ is elementary iff $r = \{ (h(X), h(\alpha)) : h \in End(S) \}$ for some $X$ and for some $\alpha$. In turn, $C^A_R$ stands for the consequence operation determined by $A$ ($A \subseteq S$) and by the set of the rules $R$. For simplicity the symbol $C_R$ will be used instead of $C^A_R$. Two particular sets of rules will be used: $MP$ – the set which contains only the modus ponens and the Gödel’s rule. $CL$ stands for the set of all classical tautologies. $I$ is the set of all theses of the intuitionistic logic and $J$ denotes the set of all theses of the Johansson’s minimal logic. $JG$ is the set of all
theses of the weakest logic, in the family of logics containing Johansson’s minimal logic, for which Glivenko’s theorem holds (cf. [7], p. 46). $S4$ is the set of all theses of the well-known modal logic. We say that $C_1$ is $C_2$ proper ($C_1 \in P(C_2)$) if and only if: $C_1(X) = S \Rightarrow h(X) \not\subseteq C_2(\emptyset)$, for every $X \subset S$ and for every $h \in \text{End}(S)$.

1. Structural completeness versus Tarski’s property

She notion of structural completeness has been introduced by W. A. Pogorzelski [6] and it reads as follows: the consequence $C$ is structurally complete ($C \in SCpl$) iff every structural and permissible rule of $C$ is derivable in it.

If the set $X$ is the only one Lindenbaum’s extension of the consistent consequence $C_1$ then we say that the consequence $C_1$ has Tarski’s property in relation to $X$ (cf. [8], [1]).

**Lemma 1.1.** For every consequences $C_1, C_2, C_3, C_4$ such that $C_4 \leq C_1 \leq C_2$, $C_3 \in P(C_2)$ and $C_2(X) = C_3(C_4(X))$ for every $X \subset S$ we get: $C_1 \in SCpl$ \iff $L(C_1) = L(C_2)$.

**Proof.** Let $C_1 \in SCpl$. Since $C_1 \leq C_2, C_1(\emptyset) = S$ and $C_2(X \cup \{\alpha\}) = S$ for every $\alpha \not\in X$. Therefore, what we need to prove is $C_2(\emptyset) \not\subseteq S$.

Suppose, to the contrary, that $C_3(C_4(X)) = S$. Since $C_3 \in P(C_2)$ then we get $h(C_4(X)) \not\subseteq C_2(\emptyset)$ for every $h \in \text{End}(S)$. Consider the rule $r_1 = \{\langle h(C_4(X)), \beta \rangle : \beta \in S \text{ and } h \in \text{End}(S)\}$. The rule $r_1$ is structural and permissible in $C_1$ but it is not derivable in $C_1$, so $C_1 \not\subseteq SCpl$ which is impossible. Therefore $L(C_1) \subset L(C_2)$. Now let $X \subset L(C_2)$. We have $C_2(X \cup \{\alpha\}) = S$ for every $\alpha \not\in X$. Since $C_1 \leq C_2$, then $C_1(X) \not\subseteq S$. Assume to the contrary that for some $\alpha \in S \setminus X$ we have $C_1(X \cup \{\alpha\}) \not\subseteq S$. In this case we consider the rule $r_2 = \{\langle h(C_4(X \cup \{\alpha\}), \beta \rangle : \beta \in S \text{ and } h \in \text{End}(S)\}$. By assumption of the lemma we get $C_3(C_4(X \cup \{\alpha\})) = S$. Since $C_3 \in P(C_2)$ then we obtain: $h(C_4(X \cup \{\alpha\})) \not\subseteq C_2(\emptyset)$ for every $h \in \text{End}(S)$. Thus the structural rule $r_2$ is permissible in $C_1$. By assumption we have $C_4 \leq C_1$, so $C_1(C_4(X \cup \{\alpha\})) \not\subseteq S$ and therefore $r_2$ is not derivable in $C_1$ which contradicts the assumption.

Putting $C_3 = C_2$ and $C_4 = IdId(X) = X$ for every $X \subset S$ we get:

**Corollary 1.2.** Let $C_1 \leq C_2$ and $C_2 \in P(C_2)$. If $C_1 \in SCpl$, then $L(C_1) = L(C_2)$. 


The assumption $C_2 \in P(C_2)$ is fulfilled, for example, when $C_2 = \overrightarrow{U}$ and $U$ is a functionally complete matrix. If we add to the assumptions of Lemma 1.1 that $C_2$ is Post-complete (i.e. $C_2(\{\alpha\}) = S$ for every $\alpha \in S - C_2(\emptyset)$), then $L(C_1) = \{C_2(\emptyset)\}$.

The assumption of Lemma 1.1 saying that $C_1 \in SCpl$, cannot be omitted. For example, if $C_3 = C_2 = C_{MP}CL$, $C_4 = Id$, then $C_1 \notin SCpl$ and $L(C_1) \neq L(C_2)$ because $S - \{p_1\} \notin L(C_1) - L(C_2)$. By Lemma 1.1 we get:

**Theorem 1.3.** If $C$ is structurally complete, $C \leq C$ and $C(\overrightarrow{U}) = CL$, then $C$ has Tarski’s property (i.e. $L(C) = \{CL\}$).

This theorem can be generalized. For instance $\overrightarrow{U} \circ Sb$ is Post-complete for any functionally complete matrix corresponding to $S$, thus if $C$ is structurally complete, $Sb \leq C$ and $C(\overrightarrow{U}(\emptyset)) = \overrightarrow{U}(\emptyset)$, then by Lemma 1.1 putting $C_1 = C$, $C_2 = \overrightarrow{U} \circ Sb$, $C_3 = \overrightarrow{U}$ and $C_4 = Sb$ we get:

$L(C) = \{\overrightarrow{U}(\emptyset)\}$.

These results give the connection between $SCpl$ and Tarski’s property.

2. Structural completeness and finite model property

Let $K_C = \{U : \overrightarrow{U}$ is a finite matrix corresponding to $S$ such $C \leq \overrightarrow{U}\}$ that $C \leq \overrightarrow{U}$. By a theory of $C$ we mean any set $X$ of formulas satisfying $X = C(X)$. We say that $X$ has the finite model property corresponding to $C$ (has fmp($C$)) iff $X = \bigcap \{E(U) : X \subset E(U) \land U \in K_C\}$.

**Theorem 2.1.** If $C \circ Sb$ is a structurally complete consequence operation and $C \circ Sb(\emptyset)$ has fmp($C_0$), then every theory of $C \circ Sb$ has fmp($C_0$).

**Proof.** Let $\overrightarrow{K}_{C_0}$ be the consequence operation determined by $K_{C_0}$ (see [10]), i.e. $\overrightarrow{K}_{C_0}(X) = \bigcap \{\overrightarrow{U}(X) : U \in K_{C_0}\}$. Since $C \circ Sb(\emptyset)$ has fmp($C_0$) then we have $C \circ Sb(\emptyset) = \overrightarrow{K}_{C_0}(\emptyset)$. Thus, each rule permissible for $\overrightarrow{K}_{C_0}$ is also permissible for $C \circ Sb$. But $\overrightarrow{K}_{C_0}$ is structural, $C \circ Sb$ is structurally complete and hence $\overrightarrow{K}_{C_0} \leq C \circ Sb$. Moreover, we have $\overrightarrow{K}_{C_0} \circ Sb(X) = \bigcap \{E(U) : X \subset E(U) \land U \in K_{C_0}\}$ which follows from the equality $U(\overrightarrow{Sb}(X)) = \bigcap \{E(U_1) : U_1 \subseteq U\}$ (comp. [9]) and from that $\overrightarrow{U} \leq U_1$ whenever $U_1 \leq U$. We conclude that for every theory $X$ of $C \circ Sb$, ...
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\[ X = \overline{K_{C_\circ}(Sb(X))} = \bigcap\{E(U) : X \subset E(U) \land U \in K_{C_\circ}\}. \]

Therefore every theory \( X \) of \( C \circ Sb \) has \( fmp(C_\circ) \).

We know (cf. [2],[5]) that there are theories of \( C^I_{MP} \circ Sb \) which lack \( fmp(C_{MP}) \), but for example \( C^J_{MP}(\emptyset) \) and \( C^I_{MP}(\emptyset) \) have \( fmp(C_{MP}) \) (cf. [7]). Moreover, R. I. Goldblatt proved that \( C^J_{MP} \) has also \( fmp(C_{MP}) \).

Thus, by Theorem 2.1 a consequence \( C^A_{MP} \) of \( C^I_{MP} \) has \( fmp(C_{MP}) \) and there exists a set \( X \) being a theory of \( C^A_{MP} \circ Sb \) which lacks \( fmp(C_{MP}) \), then \( C^A_{MP} \circ Sb \) is not SCpl. (For example \( C^J_{MP} \circ Sb, C^I_{MP} \circ Sb, C^G_{MP} \circ Sb \) are not structurally complete).

By virtue of [3], we get a theory \( X \) of \( C^{S4}_{MP} \circ Sb \) which has not \( fmp(C_{MP}) \). Therefore, by Theorem 2.1, \( C^{S4}_{MP} \circ Sb \) is not structurally complete if \( A \subset X \) and \( C^{A}_{MP}(\emptyset) \) has \( fmp(C_{MP}) \). Hence for example \( C^{S4}_{MP} \circ Sb \not\in SCpl \).

Since \( C^{A}_{MP} \circ Sb \not\in SCpl \) and there are theories of \( C^{A}_{MP} \circ Sb \) which lack \( fmp(C_{MP}) \) then the assumption saying that \( C \circ Sb \in SCpl \) is essential.

From Theorem 2.1 we get that the implication of Lemma 1.1 is not reversible. Indeed, \( C^I_{MP} \) is not SCpl. On the other hand it is easy to prove that \( L(C^I_{MP}) = L(C^{CL}_{MP}) \).

We say that \( C \) is finitely based by means of elementary rules iff \( C = C_R \) for some finite set \( R \) of finitary and elementary rules. Let \( C \) be the structural consequence operation finitely based by means of elementary rules and \( X \) let be a theory of \( C \circ Sb \). We say that \( X \) is finitely axiomatizable iff \( X = C \circ Sb(X_f) \) for some finite set \( X_f \subset X \). Theorem 2.1 and Harrop’s theorem on decidability (cf. [4]) yield:

**Corollary 2.2.** Let \( C \) be a structural consequence operation finitely based by means of elementary rules and such that \( C(\emptyset) \) has \( fmp(C) \). If \( C \circ Sb \in SCpl \) then every finitely axiomatizable theory of \( C \circ Sb \) is decidable.

References


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