Steve Giambrone

GENTZEN SYSTEMS AND DECISION PROCEDURES
FOR RELEVANT LOGICS
(Abstract)\(^1\)

In [5] and [6] Aleksander Kron claims to give (1) subscripted Gentzen formulations for the relevant logics \(R_+, T_+, RW_+, TW_+\), which he calls \(G_2R_+, G_2T_+, GR_+, GT_+\), respectively; and (2) a proof of decidability for the latter two systems, and thus (on the basis of (1)) for \(RW_+\) and \(TW_+\). However, his proofs for (1) and (2) are unsound, in spite of the correction in [7]. In discussing subscripted Gentzen systems, \(X', Y', W'\) and \(W'\) are used as variables ranging over sequences of prefixed formulae, and \(a', b', c', \ldots\) stand for finite subsets of \(\{1, 2, 3, \ldots\}\).

With respect to (1) his proffered proofs for the admissibility of Cut are wrong. All of them fail at the following case: Assume that (i) \(X \vdash x (A \rightarrow B)\) is derivable following from (iii) \(X, aA \vdash a\cup x B\), and that (ii) \(Y, W, x (A \rightarrow B) \vdash c C\) is also derivable and follows from (iv) \(Y \vdash y A\) and (v) \(W, x \cup y B \vdash c C\). It is proposed that we show (vi) \(Y, W, X \vdash c C\) by first changing \(a\) to \(y\) in (iii) then applying Cut to it and (iv) to obtain (vii) \(X, Y \vdash x \cup y B\). One can then apply Cut to (vii) and (v) to obtain (viii) \(W, X, Y \vdash c C\) from which the desired (vi) would follows by permutation. However, the subscript \(a\) can not always be changed to \(y\), crucially when \(a\) is a singleton but \(y\) is not. Thus his argument fails.

---

\(^1\)The results announced here are all contained in [4]. However, many of the results will be published separately in papers now in preparation.

\(^2\)These systems are the positive subsystems of \(R\) and \(T\) with and without the condition axiom: \((A \rightarrow A \rightarrow B) \rightarrow A \rightarrow B\), respectively. Appropriate axiomatic formulations can be gotten from [1], pp. 340–41. Kron uses the notation \(R_+^W\) and \(T_+^W\), but we prefer the conventions of R. K. Meyer for brevity.
And indeed it is demonstrable that neither Cut nor Modus Ponens is admissible in $G_2R_+$ and $GR_+ - W$, and that Modus Ponens is not admissible in $G_2T_+$ and $GT_+ - W$. For if they were, the following theorem of $TW_+$ (and thus of the other systems) would be provable in them:

$$(vi) \ (p \rightarrow q \lor r) \land (p \rightarrow p) \rightarrow (p \land q \rightarrow r) \land (r \rightarrow r) \rightarrow .p \rightarrow r.$$  

But $GT_+ - W$ and $GR_+ - W$ are decidable, and it is easy to show that (vi) cannot be proved therein. And although $G_2R_+$ and $D_2T_+$ are not known to be decidable, one can demonstrate that (vi), at least, is not derivable therein.

Now with respect to (2), the putative proof in [6] that $GT_+ - W$ and $GR_+ - W$ are decidable rests upon Theorem 6.5 (p. 75) which is simply false. Put simply the theorem says that if $aA$ and $bB$ occur in the antecedent of a sequent (in some derivation) and $a \cap b = \emptyset$, and further, if $a' A'$ and $b' B'$ (likewise occurring in the antecedent of some (one) sequent in that derivation) are “descendants” of $aA$ and $bB$, respectively, then $a' \cap b' = \emptyset$. But the author ignores the fact that such $aA$ and $bB$ may have the exact same descendant, as following (segment of) a derivation clearly demonstrates:

\[
\begin{align*}
&\text{IR} \quad \frac{\[2\] (A \rightarrow B),\{3\} (B \rightarrow C),\{4\} A \vdash \{2,3,4\} C}{(A \rightarrow B),\{3\} (B \rightarrow C) \vdash A \rightarrow C} \\
&\text{IL} \quad \frac{\{2\} (A \rightarrow B), \vdash \{2\} (B \rightarrow C) \vdash (A \rightarrow C)}{(A \rightarrow B),\{1\} \vdash (B \rightarrow C) \vdash (A \rightarrow C) \vdash (D \vdash \{1,2\} D)}
\end{align*}
\]

$\{1\} (B \rightarrow C \rightarrow A \rightarrow C \rightarrow D)$ is a descendant of both $\{3\} (B \rightarrow C)$ and $\{4\} A$, but fails to satisfy the claim of the theorem.

The two systems are indeed decidable, but given that Cut and/or Modus Ponens are not admissible, as the case may be, they are too weak for this fact to be interesting. We should also note that the putative independent proof that Modus Ponens is admissible in $GT_+ - W$ ([6], pp. 72–73) is also unsound. For (i) on page 72 is false. It conflates $B$’s not being deleted from $J$ with its being retained, that is, being a member of $K$ – which is to ignore the possibility that $B$ is not deleted from $J$ because it wasn’t there to begin with. (The notation is again that of Kron.)

---

3This observation was originally due to R. K. Meyer.
Whether suitable Cut-free subscripted Gentzen systems can be formulated for $R_+ , T_+$ and the corresponding contraction-less systems remains an open problem. However, we have succeeded in showing that a modification of $G_1 R_+$ from [6] is equivalent in an appropriate sense to the positive semilattice relevant logic of [3], which we call $UR_+$. We will call this Gentzen system $GR_+$. It comes from $G_1 R_+$ simply by disallowing sequents that are empty on the left. (A stipulation to that effect on IR suffices). Then

**Theorem 1.** $\{1\} t_A \vdash \{1\} A$ is derivable in $GR_+$ iff $\models_{UR_+} A$, where $\{1\} t_A$ is $\{1\} (p^1 \rightarrow p^1), \ldots , \{1\} (p^n \rightarrow p^n)$ and $p^1, \ldots , p^n$ are all of the (distinct) propositional variables that occur in $A$.

The left to right part of the theorem is proved by giving a translation from sequents of $GR_+$ into conditional statements about $UR_+$ models, which yields the result that if $\{1\} t_A \vdash \{1\} A$ is $GR_+$ derivable, then $A$ is valid in the semilattice semantics.

For right to left, we give a procedure for translating nodes of proofs in the natural deduction system [2] into $GR_+$ derivable sequents. This strategy is adopted to avoid dealing directly with $UR_+$’s ungainly special rule ([3], p. 234). Obviously this proof rests very heavily on the results of [2] and [3].

With respect to the decidability question for $RW_+$ and $TW_+$, we give an answer in the affirmative. For this purpose we formulate Dunn-style Gentzen systems, $LR'_W$ and $LTW'_T$, similar to $LR_+$ of [1]. Cut theorems are provided and suitable equivalences are proved. Decidability is then proved via König’s Lemma.

As usual, the finite fork property is guaranteed by the rules. The strategy for proving finite branch is essentially that of playing off the “complexity” of the conclusion against the length of the branch. For this we develop the following notions, for $W, X$ and $Y$ as arbitrary structures (i.e., antecedents in the language of [1]).

---

4The reason for amending $G_1 R_+$ is that I can find no direct method of proving cut admissible. Kron’s own “proof” breaks down exactly on cases for which he claims (p. 400) that “the proof is easy”. The same difficulty crops up for his cut theorem for $GR_+ - W$, and in his putative proof that modus ponens is admissible in $G_1 T_+$ and $G_2 T_+$.

5We even provide formulations that allowed to be empty on the left, solving an annoying problem for relevant systems with extensional connectives.
Definition 1. \( E \)-reduct of \( X(Er(X)) \)
1. \( Er(A) = A \);
2. \( Er(I(X_1, \ldots, X_n)) = I(Er(X_1), \ldots, Er(X_n)) \);
3. \( Er(E(X_1, \ldots, X_n)) = E(Y_1, \ldots, Y_m) \), where \( E(Y_1, \ldots, Y_m) \) is the result of deleting for each \( 1 \leq i \leq n \) all but two occurrences of \( ER(X_i) \) from \( E(Er(X_1), \ldots, Er(X_n)) \), provided no \( X_i \) is an \( E \)-sequence.
4. \( Er(E(X_1, \ldots, E(W_1, \ldots, W_m), \ldots, X_n) = Er(E(X_1, \ldots, W_1, \ldots, W_m, \ldots, X_n)) \).

And we say that \( X \) and \( X \vdash A \) are \( E \)-reduced just in case \( X \) is \( Er(X) \). Finally, we say that a derivation is \( E \)-reduced just in case each sequent in it is. Next, for the appropriate notion of complexity.

Definition 2. Degree of structure \( X(deg(X)) \):
1. \( deg(A) \) is the total number of occurrences of \( \rightarrow \) and \( \circ \) in \( A \).
2. \( deg(I(X_1, \ldots, X_n)) = deg(X_1) + \ldots + deg(X_n) + (n - 1) \).
3. \( deg(E(X_1, \ldots, X_n)) = \max\{deg(X_1), \ldots, deg(X_n)\} \).

And the degree of a sequent \( X \vdash A(deg(X \vdash A)) \) is \( deg(X) + deg(A) \).

With considerable tedium one can prove

Lemma 1. \( X \vdash A \) is derivable in \( LRW_t^+ (LTW_t^+) \) iff there is an \( E \)-reduced derivation of \( Er(X \vdash A) \).

This lemma now allows us to count, as recorded in

Lemma 2. For any sequent \( X \vdash A \), there are only finitely many \( E \)-reduced structures of degree \( \leq deg(X \vdash A) \) such that each wff occurring in the structure is a subwff of some wff occurring in \( X \vdash A \).

Given some of the usual straightforward properties of Gentzen systems, the finite branch property follows from Lemma 2 and

Lemma 3. If \( X \vdash B \) occurs in a derivation of \( X \vdash A \), then \( deg(Y \vdash B) \leq deg(X \vdash A) \).

So we declare

Theorem 2. \( TW_+ \) and \( RW_+ \) are decidable.

With respect to \( EW_+ \), we report the unusual situation of having a
decidable Gentzen formulation, $LEW^t_+$, which does not decide $EW_+$. For although we can show
Equivalence. $\vdash_{EW_+} A \iff X \vdash A$ is $LEW^+_t$ derivable, for some structure $X$ built up out of $t$'s.

we do not know in general which $X$ will do, if any. In effect, the system does not adequately reflect the fact that $t$ is an identity. We can give formulations of $EW_+$ that give a precise equivalence, but can not show that they are decidable. So the decision question for $EW_+$ remains open.

References