A MINIMAL EQUATIONAL BASE FOR CERTAIN VARIETIES OF BCK-ALGEBRAS

In [4] Cornish has proved that the variety of commutative BCK-algebras is 2-based and asks whether the variety of implicative BCK-algebras is also such. In this paper we show that the answer to this question is positive. Moreover we show that many of the BCK-varieties under consideration and their finitely based subvarieties have two-element equational base. For the definitions and properties of the used notions the reader is referred to [1] and [5].

We use small Latin letters for the variables and the capital (sometimes with index) for BCK-terms. In order to make our notation more readable we adopt the convention of associating to the left and ignoring the symbol of binary operation in BCK-language. Moreover we define

\[ xy^0 = x \]
\[ xy^{n+1} = xy^ny. \]

Some important varieties are the classes of all BCK-algebras (called quasicommutative) in which the following identity (with fixed non-negative integers \( m, n, k, l \)) holds

\[ (Q_{m,k}^{m,l}) \quad x(xy)^{m+1}(yx)^k = y(yx)^{n+1}(xy)^l \]
(see e.g. [7]).

Another known BCK-variety is given by

\[ (J) \quad x(x(y(xy))) = y(y(x(xy))). \]

It is obvious that the varieties of implicative, positive implicative and at last commutative BCK-algebras are subvarieties of suitable quasicommutative BCK-varieties. On can prove that they are also subvarieties of the class
of BCK-algebras with condition (J). To prove the main theorem we need some lemmas.

**Lemma 1.** The following sets of identities are equivalent

1. $x0 = x$
2. $xyx = 0$
3. $A = 0$
4. $B = 0$

and (1), (5) $uv(u(xy(xAB))(sts)) = 0$

where $u, v, x, y, s, t$ do not occur in $A$ and $B$.

**Proof.** It is easy to see that (1)–(4) imply (5). Conversely, suppose (1) and (5). First in (5) we substitute $u = xyx$, $s = xyx$, $t = xy(xAB)$ and then, also in (5), $u = xyx$, $v = xy(xAB)$, $s = x$, $t = y$. Now using (1) we obtain

$$xxyv = 0.$$ (6)

Now (2) follows from (6) and (1) with $v = 0$. Putting $u = xy(xAB)(xyx)$, $v = xyx$, $s = x$, $t = y$ in (5) and $x = xy(xAB)$, $y = xyx$, $v = xyx$ in (6) and using (1), (2) we have

$$xy(xAB) = 0.$$ (7)

Now from (7) with $x = B$, $y = 0$ and (1), (2) we obtain $B = 0$. Moreover from (1) and (2) we have successively

$$ABAB = ABAB0 = ABAB(ABA) = 0.$$

Using (1) and the above equation to (7) with $x = AB$, $y = B$ we can complete the proof of Lemma 1.

In the same way we can prove

**Lemma 2.** The identities

$$x(00^m) = x$$
$$uv(u(xy(xE0))(sts)) = 00^m$$

imply $00^m = 0$ and $E = 0$, $(u, v, x, y, s, t$ do not occur in $E)$.
By \( \omega \)-identity we mean the identity \( A = B \) in the \( BCK \)-language such that \( A, B \) have only two variables (say \( x \) and \( y \)) and \( xy = 0 = yx \) imply \( A = x, B = y \) or inversely. By \( \omega \)-variety we mean a variety of \( BCK \)-algebras in which certain fixed \( \omega \)-identity holds. In particular, each of the varieties considered above is \( \omega \)-variety.

**Theorem 1.** Each finitely based \( \omega \)-variety is 3-based, there exists an equational base of the form

\[
\begin{align*}
x_0 &= x \\
A &= 0 \\
(\omega - \text{identity}) - A_0 &= B_0.
\end{align*}
\]

**Proof.** Let \( \{A_i = B_i | i = 1, \ldots, n\} \) be an equational base for \( \omega \)-variety \( \mathbb{K} \). Put \( D_i = A_iB_i \) and \( D_{n+1} = B_iA_i \) (\( i = 1, \ldots, n \)). Of course \( D_i = 0 \) holds in \( \mathbb{K} \) for each \( j = 1, \ldots, 2n \). Moreover in \( \mathbb{K} \) \( x_0 = x \) and the suitable \( \omega \)-identity \( A_0 = B_0 \) hold. From Lemma 1 by induction we obtain the identity \( A = 0 \) such that \( x_0 = x, A = 0 \) imply \( D_j = 0 \) for each \( j \). It is easy to see that the identities \( x_0 = x, A = 0, A_0 = B_0 \) constitute an equational base for \( \mathbb{K} \).

**Remark.** The first example of the 3-base for commutative \( BCK \)-algebras was given by Pasiński and Woźniakowska in [6] and for implicative, positive implicative, quasicommutative \( BCK \)-algebras by Cornish [4], (see also [1]).

Now we are ready to prove the main result.

**Theorem 2.** Each finitely based variety in which \( (Q_{m,k}^{n,l}) \) or \( (J) \) holds is 2-based.

**Proof.** From Theorem 1 we have the equational base of the form \( x_0 = x, A = 0, (Q_{n,l}^{m,k}) \) (resp. \( (J) \)). Let the variables \( a, b, u, v, x, y, s, t \) not occur in \( A \). We define \( E = uv(u(xy)(xE0))(sts) \). We show that

\[
\begin{align*}
(1) \quad & (Q_{n,l}^{m,k}) \quad \text{(resp. } (J)) \\
(2) \quad & a(Eb) = a
\end{align*}
\]

constitute the equational base for our variety. It is clear that (2) follows from \( x_0 = x \) and \( A = 0 \). Conversely, suppose (1) and (2) hold. Then we have

\[
(3) \quad Ea = Eb.
\]
Indeed, from \((Q_{m,k}^{n,l})\) we have \(Ea(Ea(Eb))^{m+1}(Eb(Ea))^k = Eb(Eb(Ea))^{n+1} (Ea(Eb))^l\) and by induction from (2) we obtain (3). With the same substitution we can prove (3) from \((J)\).

\[
\begin{align*}
(4) & \quad Ea = E, \text{ because by (3) and (2) } Ea = E(EE) = E. \\
(5) & \quad aE = a, \text{ because by (4) and (2) } aE = a(Ea) = a. \\
(6) & \quad E = 00^j, \text{ with suitable fixed } j, \text{ is easy to obtain from (1) with } x = E \\
& \quad \text{ and } y = 0.
\end{align*}
\]

Now from the definition of \(E\), (5) and Lemma 2 we have \(A = 0\). So Theorem 2 is proved.

**Remark.** Cornish [3], [4] has proved that no non-trivial variety of the \(BCK\)-algebras is 1-based. That fact with the above theorem warrant the title of this paper.

**Corollary 1.** The varieties of implicative, positive implicative \(BCK\)-algebras are 2-based.

**Corollary 2.** Any finite \(BCK\)-algebra is 2-based.

**Proof.** Yutani [7] has proved that any finite \(BCK\)-algebra is quasicommutative, and Cornish that it is 3-based (see [2]).

**References**


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