ON THE $EK$-FRAGMENTS OF POSITIVE AND CLASSICAL PROPOSITIONAL LOGICS

Several formalizations of the (classical) propositional logic with equivalence, conjunction and falsum as the primitives are well known [1], [2], [4]. We believe that in an axiomatic system containing equivalence among its undefined connectives a very natural and powerful rule of replacement

Repl: $Eab, \phi a \Rightarrow \phi b$

(where the formula $\phi b$ results from $\phi a$ by replacing some occurrences of $a$ by those of $b$) deserves to be, as in [1], a primitive one. We present here three axiom systems with Repl as a single primitive rule: one for the $EK$-fragment of the positive (or intuitionistic) logic, and two for that of the classical propositional logic. In all the calculi considered below we have dispensed with the rule of substitution in favour of axiom schemes.

Let $CKE_H$ be a calculus determined by a single rule $MP$ and the following well-known axioms of the classical propositional logic:

$P_1 : \ Cacba$
$P_2 : \ CCaBbcCCabCac$
$P_3 : \ CCCabaa$
$P_4 : \ CKaba$
$P_5 : \ CKabb$
$P_6 : \ CCabCCacCaKbc$
$P_7 : \ CEabCab$
$P_8 : \ CEabCba$
$P_9 : \ CCabCCbaEab,$

and let $CKE_H = CKE - P3$ be the corresponding fragment of the positive logic (cf. [3]). Consider the following formulas:
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$S_1 : \ E_aE_aE_bb$
$S_2 : \ EKaaa$
$S_3 : \ EKabKba$
$S_4 : \ EKabKeKbc$
$S_5 : \ EKabEaEKabKac$
$S_1^* : \ EEEbaba$
$D : \ ECabEaKab$

Let $EK_\Pi = \{S_1, S_2, S_3, S_4, S_5, Repl\}$, and let $EK_\mathcal{H} = EK_\Pi - S_1 + S_1^*$, $EK_\mathcal{C}_\Pi = EK_\Pi + D$, $EK_\mathcal{C}_\mathcal{H} = EK_\mathcal{H} + D$. In what follows, $a$ stands for any formula, and $\Gamma$ – for any, possibly empty, set of formulas of the calculus $EK_\Pi$.

**Lemma.**
(a) $EK_\mathcal{C}_\Pi + \Gamma \vdash a$ iff $EK_\Pi + \Gamma \vdash a$,
(b) $EK_\mathcal{C}_\Pi \not\vdash Ecd$, where $c = S_1^*$, $d = P3$,
(c) $EK_\mathcal{C}_\mathcal{H} \vdash S_1$.

**Theorem.**
(a) $RK_\Pi + \Gamma \vdash a$ iff $CKE_\Pi + \Gamma \vdash a$,
(b) $EK_\mathcal{H} + \Gamma \vdash a$ iff $CKE_\mathcal{H} + \Gamma \vdash a$.

**Corollary.**
(a) In $EK_\Pi, \Gamma \vdash Kab$ iff $\Gamma \vdash a$ and $\Gamma \vdash b$,
(b) In $EK_\Pi, \Gamma \vdash Eab$ iff $\Gamma + a \vdash b$ and $\Gamma + b \vdash a$.

**Proofs.** The assertion (a) of the lemma is trivial. The other two can be checked by uninteresting direct calculation. Similarly the deductive equivalence of $EK_\Pi$ and $CKE_\Pi$ can be proved. Now it is easy to get the theorem and the corollary.

**Remark.** From the system investigated in [1] one can extract another classical $EK$-calculus, $EK_\Pi' = \{L_1, S_2, S_4, S_5, Repl\}$, where
$L_1 : \ EEabEEcbEac$
$S_5' : \ EKabEaeEaKbaKac$.

This calculus contains only four axiom schemes, but it can be further simplified: it turns out that the calculus $EK_\Pi'' = EK_\mathcal{H} - (S_3, S_5) + S_5'' = \ldots$
$EK'_H - (L, S5) + (S1^*, S5'')$, where $S5'' : EK aEbcEaEKabKca$ is deductively equivalent to $EK_H$.

**Problem.** It is highly plausible that the axiom systems $EK_H$ and $EK'_H$, as well as $EK''_H$ and $EK''_H$, are independent. Assuming this as hypothesis, find a finite set $\{A_1, \ldots, A_n\}$ of additional axiom schemes such that the system $EK_H + (A_1, \ldots, A_n)$ is independent and deductively equivalent to $EK_H$.

**References**


