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IDEALS IN $BCK$-ALGEBRAS WHICH ARE LOWER SEMILATTICES

This is an abstract of the paper presented at the seminar held by prof. A Wroński at the Jagiellonian University.

It was shown in [1] that if $X$ is a $BCK$-algebra then $(X, \leq)$ is a poset, and moreover if $X$ is a commutative $BCK$-algebra, i.e. $x \ast (x \ast y) = y \ast (y \ast x)$ holds in $X$, then $(X, \leq)$ is a lower semilattice. In this paper we consider properties of certain ideals in these $BCK$-algebras which are lower semilattices as referred to [1] and [2]. The following example shows that the class of $BCK$-algebras considered here is considerably wider than the class of commutative $BCK$-algebras.

Example. Consider the set $\{0, 1, \ldots\}$. We define

$$x \ast y = \begin{cases} 
0 & \text{if } x \leq y \\
1 & \text{if } 0 \neq y < x \\
x & \text{if } 0 = y < x
\end{cases}$$

It is easy to check that $\langle\{0, 1, \ldots\}, \ast, 0\rangle$ is a non-commutative $BCK$-algebra. $(X, \leq)$ is a lattice.

Let us recall some definitions.

A non-empty subset $A$ of a $BCK$-algebra $X$ is called an ideal iff

1. $0 \in A$;
2. $x \in A$ and $y \ast x \in A$ imply $y \in A$.

A proper ideal of a $BCK$-algebra $X$ is maximal if it is not properly contained in any proper ideal in $X$. 
Let $X$ be a BCK-algebra and $B$ a subset of $X$. By $(B)$ we denote an ideal in a BCK-algebra $X$ generated by $B$. If $B$ is finite, i.e. $B = \{b_1, \ldots, b_k\}$, we shall write $(b_1, \ldots, b_k]$ instead of $(\{b_1, \ldots, b_k\}]$.

An ideal $A$ in a BCK-algebra is called irreducible if $A = B \cap C$ implies $A = B$ or $A = C$, for ideals $B, C$. Let $X$ be a BCK-algebra which is a lower semilattice. We shall call an ideal $A$ in $X$ prime if for any elements $a, b$ of $X$ $\inf\{a, b\} \in A$ implies $a \in A$ or $b \in A$. This notion generalizes the notion of a prime ideal in a commutative BCK-algebra introduced by K. Iseki in [3].

In the sequel, by a BCK-algebra we shall mean a BCK-algebra which is a lower semilattice as a poset. We shall denote $\inf\{x, y\}$ by $x \land y$.

We have the following Lemma:

**LEMMA.** In a BCK-algebra $X$, if for some natural numbers $m$ and $n$ $a \ast^m x = a \ast^n y = 0$, then there exists a natural number $p$ such that $a \ast^p (x \land y) = 0$, where $a \ast^m x$ denotes $(\ldots (a \ast x) \ast \ldots) \ast x$, $m$-times.

The above Lemma has some interesting consequences.

**COROLLARY 1.** Let $X$ be a BCK-algebra, $P$ an ideal in $X$. Then for any $x, y \in X$, if $x \land y \in P$ then $(P \cup \{x\}] \cap (P \cup \{y\}) = P$.

**COROLLARY 2.** In a BCK-algebra $X$

$$(x \land y) \cap (y \land y) = (x \land y)$$

We shall call a non-empty subset $S$ of a BCK-algebra $X$ \land-closed iff for any $x, y \in X, x, y \in S$ implies $x \land y \in S$.

**COROLLARY 3.** Let $X$ be a BCK-algebra, $S$ a non-empty \land-closed subset of $X$ such that $0 \notin S$. Then there exists a maximal ideal $P$ in the set of all ideals $I$ in $X$ such that $I \cap S = \emptyset$, moreover $P$ is a prime ideal.

**COROLLARY 4.** If $P$ is a maximal ideal in a BCK-algebra $X$ then $P$ is a prime ideal.

Using Corollaries one can prove the following two Theorems:

**THEOREM 1.** In a BCK-algebra, the following conditions are equivalent

(i) $P$ is an irreducible ideal;
(ii) $P$ is a prime ideal;
(iii) for any ideals $A, B, A \cap B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

**Theorem 2.** The lattice of all ideal of a BCK-algebra $X$ is distributive.

**Remark.** Theorem 2 holds true for an arbitrary BCK-algebra.

**References**


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