Bulletin of the **S**ection of **L**ogic
Volume 10/1 (1981), pp. 48–50

reedition 2009 [original edition, pp. 48–51]

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IDEALS IN BCK-ALGEBRAS WHICH ARE LOWER SEMILATTICES

This is an abstract of the paper presented at the seminar held by prof. A Wroński at the Jagiellonian University.

It was shown in [1] that if X is a BCK-algebra then $(X, \leq)$ is a poset, and moreover if X is a commutative BCK-algebra, i.e. $x * (x * y) = y * (y * x)$ holds in X, then $(X, \leq)$ is a lower semilattice. In this paper we consider properties of certain ideals in these BCK-algebras which are lower semilattices as referred to [1] and [2]. The following example shows that the class of BCK-algebras considered here is considerably wider than the class of commulative BCK-algebras.

Example. Consider the set $\{0, 1, \ldots\}$. We define

$$x * y = \begin{cases} 0 & \text{if} \quad x \leq y \\ 1 & \text{if} \quad 0 \neq y < x \\ x & \text{if} \quad 0 = y < x \end{cases}$$

It is easy to check that $\langle\{0, 1, \ldots\}, *, 0\rangle$ is a non-commutative BCK-algebra. $(X, \leq)$ is a lattice.

Let us recall some definitions.

A non-empty subset A of a BCK-algebra X is called an ideal iff

1. $0 \in A$;
2. $x \in A$ and $y * x \in A$ imply $y \in A$.

A proper ideal of a BCK-algebra X is maximal if it is not properly contained in any proper ideal in X.

Let X be a BCK-algebra and B a subset of X. By $(B]$ we denote an ideal in a BCK-algebra X generated by B. If B is finite, i.e. $B = \{b_1, \ldots, b_k\}$, we shall write $(b_1, \ldots, b_k]$ instead of $(\{b_1, \ldots, b_k\}]$.

An ideal A in a BCK-algebra is called irreducible if $A = B \cap C$ implies $A = B$ or $A = C$, for ideals B, C.

Let X be a BCK-algebra which is a lower semilattice. We shall call an ideal A in X prime if for any elements a, b of X $inf\{a, b\} \in A$ implies $a \in A$ or $b \in A$. This notion generalizes the notion of a prime ideal in a commutative BCK-algebra introduced by K. Iseki in [3].

In the sequel, by a BCK-algebra we shall mean a BCK-algebra which is a lower semilattice as a poset. We shall denote $inf\{x, y\}$ by $x \wedge y$.

We have the following Lemma:

Lemma. *In a BCK-algebra X, if for some natural numbers m and n $a *^m x = a *^n y = 0$, then there exists a natural number p such that $a *^p (x \wedge y) = 0$, where $a *^m x$ denotes $(\ldots(a * \underbrace{x) * \ldots) * x}_{m-times}$.*

The above Lemma has some interesting consequences.

Corollary 1. *Let X be a BCK-algebra, P an ideal in X. Then for any $x, y \in X$, if $x \wedge y \in P$ then $(P \cup \{x\}] \cap (P \cup \{y\}] = P$.*

Corollary 2. *In a BCK-algebra X*

$$(x] \cap (y] = (x \wedge y]$$

We shall call a non-empty subset S of a BCK-algebra X $\wedge$-closed iff for any $x, y \in X, x, y \in S$ implies $X \wedge y \in S$.

Corollary 3. *Let X be a BCK-algebra, S a non-empty $\wedge$-closed subset of X such that $0 \notin S$. Then there exists a maximal ideal P in the set of all ideals I in X such that $I \cap S = \emptyset$, moreover P is a prime ideal.*

Corollary 4. *If P is a maximal ideal in a BCK-algebra X then P is a prime ideal.*

Using Corollaries one can prove the following two Theorems:

Theorem 1. *In a BCK-algebra, the following conditions are equivalent*

(i) P is an irreducible ideal;

(ii) P is a prime ideal;
(iii) for any ideals $A, B, A \cap B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

THEOREM 2. *The lattice of all ideal of a BCK-algebra X is distributive.*

REMARK. Theorem 2 holds true for an arbitrary BCK-algebra.

References

[1] K. Iseki, S. Tanaka, *An introduction to the theory of BCK-algebras*, **Mathematica Japonica** 23 (1978), pp. 1–26.

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[3] K. Iseki, *On some ideals in BCK-algebras*, **Mathematics Seminar Notes** 3 (1975), pp. 65–70.

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