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KRIPKE-STYLE SEMANTICS FOR JAŚKOWSKI'S SYSTEM Q_f

Classical logic, intuitionism, relevant logics and many other systems try to express implication as an entailment. The Jaśkowski system Q_f (cf. [2]) describes implication in connection with causality. Syntactic properties of Q_f have been examined by Pieczkowski [4], [5]. The semantic (Kripke-style) characterization of Q_f is the aim of this paper.

The set FOR of (well-formed) formulae ($wffs$) is formed from the set At of sentential variables by using one unary connective $\neg$ and two binary connectives $\wedge$, $\forall_f$ in the usual way. The expressions $\neg\alpha, \alpha \wedge \beta$ and $[\forall_f \alpha]\beta$ should be read "no α", "α and β" and "*beta* for all factors of α", respectively. The variables can be treated, similarly to Heyting's ([1]) and Jaśkowski's ([3]) methods, as dependent sentential variables, i.e. predicate variables but with undetermined arity. Then formulas can be identified with sentential functions. Relevant arguments of a formula α, on which α really depends, are called *factors of* α.

Let $t_n, n \in \omega$, being an interpretation of the set FOR into the set of $wffs$ of the first-order predicate calculus PC, be defined as follows:

1) $t_n(p) = P(x_1, \ldots, x_n)$ where P is an n-ary predicate symbol and $(x_1 \ldots x_n)$ is fixed for all $p \in At$,
2) $t_n(\neg\alpha) = \neg t_n(\alpha)$,
3) $t_n(\alpha \wedge \beta) = t_n(\alpha) \wedge t_n(\beta)$,
4) $t_n([\forall_f \alpha]\beta) = t_n(\beta) \wedge \bigwedge_{\{k_1 \ldots k_l\} \subseteq \{1 \ldots n\}} \{(\exists x_1 \ldots x_n(\exists x_{k_1} t_n(\alpha) \wedge \exists x_{k_1} \neg t_n(\alpha)) \wedge \ldots \wedge \exists x_1 \ldots x_n(\exists x_{k_l} t_n(\alpha) \wedge \exists x_{k_l} \neg t_n(\alpha))] \rightarrow \forall x_{k_1} \ldots x_{k_l} t_n(\beta)\}$.

The interpretation t_n takes $wffs$ from FOR to the set of $wffs$ of the

n-homogeneous predicate calculus (see [5]).

The system Q_f is defined in the following way:

DEFINITION 1. For all $\alpha \in FOR$:

$$\alpha \in Q_f \Leftrightarrow \forall n \in \omega : t_n(\alpha) \in PC.$$

Let $\mathcal{F}_i = \langle W_i, R_i \rangle$, $1 \leq i \leq n$, be any Kripke-frames. Relations $\widehat{R}_1, \ldots, \widehat{R}_n$ are defined on the set $W_1 \times \ldots \times W_n$ by the following conditions:

$$(x_1 \ldots x_n)\widehat{R}_i(y_1 \ldots y_n) \Leftrightarrow x_i R_i y_i \wedge \forall j \neq i : x_j = y_j.$$

We introduce the following notations:

$$\mathcal{F} = \mathcal{F}_1 \times \ldots \times \mathcal{F}_n = \langle W_1 \times \ldots \times W_n, \widehat{R}_1, \ldots, \widehat{R}_n \rangle,$$

$$\mathcal{M}_n = \langle \mathcal{F}, \varphi \rangle \text{ where } \varphi : At \to W_1 \times \ldots \times W_n.$$

By an *evaluation* $\models$ we understand a 3-ary relation between a model $\mathcal{M}_n$, a world $\overline{w} = (w_1 \ldots w_n) \in W_1 \times \ldots \times W_n$ and $\alpha \in FOR$ satisfying (1)-(4) below:

(1) $\mathcal{M}_n \models p[\overline{w}] \Leftrightarrow \overline{w} \in \varphi(p)$ for $p \in At$,
(2) $\mathcal{M}_n \models \neg\alpha[\overline{w}] \Leftrightarrow \mathcal{M}_n \not\models \alpha[\overline{w}] \Leftrightarrow \neg(\mathcal{M}_n \models \alpha[\overline{w}])$,
(3) $\mathcal{M}_n \models \alpha \wedge \beta[\overline{w}] \Leftrightarrow \mathcal{M}_n \models \alpha[\overline{w}] \wedge \mathcal{M}_n \models \beta[\overline{w}]$,
(4) $\mathcal{M}_n \models [\forall_f \alpha]\beta[\overline{w}] \Leftrightarrow \mathcal{M}_n \models \beta[\overline{w}] \wedge \bigwedge_{k_1,\ldots,k_l=1,\ldots,n} \{\mathcal{M}_n \models f(k_1 \ldots k_l, \alpha)[\overline{w}] \Rightarrow$
$[\forall x_{k_1} \ldots x_{k_l} : \overline{w}\widehat{R}_{k_1} \circ \ldots \circ \widehat{R}_{k_l}\overline{w}(w_{k_1}/x_{k_1} \ldots w_{k_l}/x_{k_l}) \Rightarrow$
$\mathcal{M}_n \models \beta[\overline{w}(w_{k_1}/x_{k_1} \ldots w_{k_l}/x_{k_l})]$

where

I) $\mathcal{M}_n \models f(i_1 \ldots i_l, \alpha)[\overline{w}] \Leftrightarrow \mathcal{M}_n \models f(i_1, \alpha)[\overline{w}] \wedge \ldots \wedge \ldots \wedge \mathcal{M}_n \models f(i_l, \alpha)[\overline{w}]$

II) $\mathcal{M}_n \models f(i, \alpha)[\overline{w}] \Leftrightarrow (\exists x_i : \overline{w}\widehat{R}_i\overline{w}(w_i/x_i) \wedge \mathcal{M}_n \models \alpha[\overline{w}(\ldots)]) \wedge (\exists x_i : \overline{w}\widehat{R}_i\overline{w}(\ldots) \wedge \mathcal{M}_n \not\models \alpha[\overline{w}(\ldots)])$.

Validity of α in a model $\mathcal{M}_n(\mathcal{M}_n \models \alpha)$, on a frame $\mathcal{F}(\mathcal{F} \models \alpha)$ and on a class $\mathcal{K}$ of frames $(\mathcal{K} \models \alpha)$ is defined as usual.

Let L be any normal logic. We introduce some more notations:

$\mathcal{K}_L^n = \{\mathcal{F}_1 \times \ldots \times \mathcal{F}_n; \mathcal{F}_1 \models L, \ldots, \mathcal{F}_n \models L\}$,

$F_L^n = \{\alpha \in FOR; \mathcal{K}_L^n \models \alpha\}$,
$F_L = \bigcap_{n \in \omega} F_L^n$.

THEOREM 2. $Q_f = F_{S5}$.

PROOF. Let $\mathcal{K}^n$ be the class of all frames $\mathcal{G} = \langle W^n, \widehat{T}_1, \ldots, \widehat{T}_n \rangle$ such that relations $\widehat{T}_i, 1 \leq i \leq n$, are defined as follows:

$$(w_1 \ldots w_n)\widehat{T}_i(v_1 \ldots v_n) \Leftrightarrow \forall i \neq j : w_j = v_j.$$

First of all, an easy induction proves that for any $\alpha \in FOR$:

(A) $t_n(\alpha) \in PC \Leftrightarrow \mathcal{K}^n \models \alpha,\ n \in \omega$.

Next we show that for any $\alpha \in FOR$:

(B) $\mathcal{K}^n \models \alpha \Leftrightarrow \mathcal{K}_{S5}^n \models \alpha;\ n \in \omega$.

(a) Since $\mathcal{K}^n \subseteq \mathcal{K}_{S5}^n$, we have $\mathcal{K}_{S5}^n \models \alpha$ only if $\mathcal{K}^n \models \alpha$.

(b) Let $\mathcal{K}_{S5}^n \not\models \alpha$.

There exists $\mathcal{F} = \langle W_1 \times \ldots \times W_n, \widehat{R}_1, \ldots, \widehat{R}_n \rangle \in \mathcal{K}_{S5}^n$, $\varphi : At \to_2 W_1 \times \ldots \times W_n$ and $\overline{x}' = (x_1' \ldots x_n')$ such that

$$\mathcal{M}_n = \langle \mathcal{F}, \varphi \rangle \not\models \alpha[\overline{x}'].$$

Let $\mathcal{F}'$ be the subframe of $\mathcal{F}$ generated by $\overline{x}'$, i.e.

$$\mathcal{F}' = \langle N, \widehat{S}_1, \ldots, \widehat{S}_n \rangle$$

where $N = \{y \in W_1 \times \ldots \times W_n; \exists i_1 \ldots i_l \leq n : \overline{x}\widehat{R}_{i_1} \circ \ldots \circ \widehat{R}_{i_l}\overline{y}\} = (W_1 \times \ldots \times W_n)(x_1' \ldots x_n')$ and $\widehat{S}_i = \widehat{R}_{i/N}$ for $1 \leq i \leq n$.
Let $\mathcal{M}'$ be $\langle \mathcal{F}', \varphi' \rangle$ where $\varphi'(p) = \varphi(p) \cap N$.
Then

$$\mathcal{M}' \not\models \alpha[\overline{x}'].$$

Let $W^{x_i'} = \{y \in W_i; \forall \overline{z} \in W_1 \times \ldots \times W_n : \overline{z}(z_i/x_i')\widehat{R}_i\overline{z}(z_i/y)$ and $W^* = \bigcup_{i=1}^{n} W_i^{x_i'}$.
Note that $N = W_1^{x_1'} \times \ldots \times W_n^{x_n'}$.

We choose one element a_i from every $W_i^{x_i'}$ and define the projections $\Pi_i : W^* \to W_i^{x_i'}$ for $1 \leq i \leq n$ as follows:

$$\Pi_i(w) = \begin{cases} w & \text{for } w \in W_i^{x'_i} \\ a_i & \text{otherwise.} \end{cases}$$

Furthermore, $\varphi^*(p) =^{df} \{(x_1 \ldots x_n) \in (W^*)^n;\ (\Pi_1 x_1 \ldots \Pi_n x_n) \in \varphi'(p)$. The structure $\langle (W^*)^n, \widehat{T}_1, \ldots, \widehat{T}_n \langle$, where $(w_1 \ldots w_n)\widehat{T}_i(v_1 \ldots v_n) \Leftrightarrow \forall i \neq j : w_j = v_j$, is a frame from $\mathcal{K}^n$.

We show that for any $\beta \in FOR$ the following equivalence holds:

(c) $\mathcal{M}^* = \langle (W^*)^n, \widehat{T}_1, \ldots, \widehat{T}_n, \varphi^* \rangle \models \beta[\overline{x}] \Leftrightarrow \mathcal{M}' \models \beta[\Pi\overline{x}]$.

It is easy to show that all $\beta \in At$ fulfil (c) and if (c) is valid for β, γ, then it is also valid for $\neg\beta, \beta \wedge \gamma$. It remains to show that by this assumption (c) holds for $[\forall_f \beta]\gamma$. Let

$$\forall y \in W^* : \mathcal{M}^* \models \beta[\overline{x}(x_i/y)].$$

The induction hypothesis guarantees that

$$\forall y \in W^* : \mathcal{M} \models \beta[\Pi\overline{x}(\Pi_i x_i/\Pi_i y)].$$

This is equivalent to

$$\forall y \in W_i^{x'_i} : \mathcal{M}' \models \beta[\Pi\overline{x}(\Pi_i x_i/y)].$$

¿From clause (4) in the definition of validity and from the above it follows that

$$\mathcal{M}^* \models [\forall_f \beta]\gamma[\overline{x}] \Leftrightarrow \mathcal{M}' \models [\forall_f \beta]\gamma[\Pi\overline{x}].$$

Whence we have proved (c).

Note that $(x'_1, \ldots, x'_n)$ is a fixpoint for Π, i.e.

$$\Pi\overline{x}' = (\Pi_1 x'_1, \ldots, \Pi_n x'_n) = \overline{x}'.$$

Therefore	$\mathcal{M}^* \models \beta(\overline{x}'] \Leftrightarrow \mathcal{M}' \models \beta[\overline{x}']$
and so	$\mathcal{M}^* \not\models \alpha[\overline{x}']$.
Whence	$\mathcal{K}^n \not\models \alpha$.

So (b) has been proved.

The equivalence (B) holds by (a) and (b).

It follows from (A), (B) and Definition 1 that for all $\alpha \in FOR$:

$$\alpha \in Q_f \Leftrightarrow \alpha \in F_{S5}.$$

QED.

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