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THE BOOLEAN ALGEBRA OF OBJECTIVES

This is the fifth and last installment in series dealing with the Wittgensteinian notion of a *situation* (cf. [1], [2]). All proofs and most lemmas have been omitted. They are contained in a comprehensive paper on the ontology of situations to be submitted to *Studia Logica*.

1. Q-spaces and V-sets

In [2] the lattice of elementary situations $\langle SE'', ;, !, \circ, \lambda\rangle$ is described. The join $x;y$ corresponds to conjunction, the meet $x!y$ has no counterpart in language; the empty situation $\underline{\circ}$ is the zero, and the impossible one λ is the unit.

Unit the product: $A \cdot B = \{x;y \in SE'' : x \in A, y \in B\}$, the power set $P(SE'')$ is a commutative semigroup; $\{\circ\}$ is the neutral element, $\emptyset$ is the annihilator. On $P(SE'')$ the relation of *involvement*:

$$A \sqsupset B \text{ iff } \bigwedge_{x \in A} \bigvee_{y \in B} y \leq x$$

is a preordering. It is monotonic with regard both to the product and to the union of SE''-sets.

The logical space SP is the product of a finite number of logical dimensions: $SP = D_1 \cdot \ldots \cdot D_n$, each dimension being a set of the atoms of SE'' (cf. [2]). Every dimension, and every product of distinct dimensions, is a *quasi-subspace* of SP, or a Q-*space* for short. The family of all Q-spaces, augmented with the *improper* Q-space $Q_\circ = \{\circ\}$, is denoted by $\underline{Q}$. Clearly all the Q-spaces are disjoint, but they may involve one another. In particular: $SP \sqsupset Q \sqsupset Q_\circ$, for any $Q \in \underline{Q}$. Moreover, the intersection of a Q-space with a *realization* R (i.e., with a maximal ideal of SE'') is never empty, being always a unit set.

In $P(SE'')$ we define the family $\underline{V}$ of *complete sets of verifiers*, or V*-sets* for short. A V-set is a non-empty SE''-set V such that for any $x, y \in SE''$, and for any $A \subset SE''$-set V such that for any $x, y \in SE''$, and for any $A \subset SE''$, we have:

(A1) $x \in V \Rightarrow (x \leq y \Rightarrow y \in V)$
(A2) $\bigvee_{Q \in \underline{Q}} (A \cdot Q \subset V) \Rightarrow A \subset V.$

For any $\underline{V}_1 \subset \underline{V}$, and for any $V_1, V_2 \in \underline{V}$, the following are theorems:

(T1) $\bigcap \underline{V}_1 \in \underline{V};\ V_1 \cap V_2 = V_1 \cdot V_2;\ V_1 \sqsupset V_2$ iff $V_1 \subset V_2$.

Thus $\langle \underline{V}, \cdot \rangle$ is a sub-semigroup of $P(SE'')$, in which the product coincides with intersection, and involvement with inclusion; the neutral element is SE'', and $\{\lambda\} = \bigcap \underline{V}$ is the annihilator.

V-equivalence of SE''-sets

The number of dimensions being finite, every chain in SE'' is finite too. Thus if $A \subset SE''$ and $x \in A$, then there is always in A a minimal element x' such that $x' \leq x$. The set of the minimal elements of A is the *minimum* of A : $Min(A) = \{x \in A :\sim \bigvee_y (y \in A \wedge y < x)\}$. The range of the map Min is the family $\underline{Min} = \{A \subset SE'' : A = Min(A)\}$ of *minimal* SE''*-sets*. Observe that

(T2) $Min(A) \sqsupset A,\ A \sqsupset Min(A)$

for any $A \subset SE''$. Thus Min is order-preserving:

(T3) $A \sqsupset B \Rightarrow Min(A) \sqsupset Min(B),$

and we have also

(T4) $A \sqsupset B \wedge B \sqsupset A$ iff $Min(A) = Min(B).$

Consequently, in $\underline{Min}$ involvement is anti-symmetric. Moreover, with $A * B = Min(A \cdot B)$, and $A + B = Min(A \cup B)$, $\langle \underline{Min}, *, +, \{\circ\}, \emptyset \rangle$ is a bounded distributive lattice, and involvement is its partial ordering.

On $P(SE'')$ we define a relation of V*-equivalence* as follows:

$$A \sim_v B \text{ iff } \bigwedge_{R \in \underline{R}} (A \cap R = \emptyset \text{ iff } B \cap R = \emptyset),$$

where $\underline{R}$ is the family of all realizations. This is a congruence with regard to the product and union of SE''-sets. Moreover,

(T5) $\quad A \sim_v Min(A)$,

and so, for any $A, B \subset SE''$, $A \sim_v B \Rightarrow Min(A) \sim_v Min(B)$. This makes it clear that $\sim_v$ is also a congruence of the lattice $\langle \underline{Min}, *, + \rangle$. Observe that for any $A \in \underline{Min}$:

(T6) $\quad A \sim_v \emptyset$ iff $(A = \emptyset$ or $A = \{\lambda\})$.

A connection between the notions of V-equivalence and V-sets is apparent in view of

(T7) $\quad V_1 \sim_v V_2$ iff $V_1 = V_2$

holding for any $V_1, V_2 \in \underline{V}$.

For any $A \in \underline{Min}, Q \in \underline{Q}$, we call $A * Q$ the *expansion* of A upon Q. And two minimal SE''-sets are Q-*equivalent* iff they have a common expansion; i.e., $A \sim_q B$ iff $\bigvee_{Q \in \underline{Q}} (A * Q = B * Q)$. Now, save for a single pair of minimal SE''-sets, $\sim_v$ coincides with $\sim_q$. The exception are $\{\lambda\}$ and $\emptyset$, for which $\sim_v$ holds, but $\sim_q$ does not. Note, moreover, that the definiens of $\sim_q$ is equivalent to $A * SP = B * SP$.

3. Objective of Tautology and Negation

Let $\underline{L}$ be a propositional language subject to classical logic. We postulate that to every proposition α of $\underline{L}$ there corresponds a complete set of its verifiers $V(\alpha)$. Moreover, the map $V : \underline{L} \to \underline{V}$ is to conform to the following four axioms:

(A3) $\quad \alpha$ is true iff $V(\alpha) \cap R_\circ = \emptyset$,

(A4) $\quad \alpha$ entails β iff $V(\alpha) \subset V(\beta)$,

(A5) $\quad V(\alpha) \cap V(\beta) \subset V(\alpha \wedge \beta)$,

(A6) $\quad y \in V(\sim \alpha)$ iff $\bigwedge_{x \in V(\alpha)} x; y = \lambda$, for any $\alpha, \beta \in \underline{L}$, any $x, y \in SE''$, with entailment interpreted as strict implication, and $R_\circ$ being the set of all real elementary situations. Thus $V(\alpha \wedge \beta) = V(\alpha) \cap V(\beta)$. This and A6 yield

(T8) $\quad x \in V(\alpha \vee \beta)$ iff $\bigwedge_{y:x \leq y \neq \lambda} \bigvee_{z:y \leq z \neq \lambda} z \in V(\alpha) \cup V(\beta)$,

which in turn yields:

(T9) $\quad V(\alpha \vee \beta) \sim_v V(\alpha) \cup V(\beta)$.

The *locus* $M(\alpha)$ of a proposition α is the set of all its maximal proper verifiers: $M(\alpha) = V(\alpha) \cap SP$. (These are possible worlds; the only *improper* verifier is λ). For loci we have clearly: $M(\alpha \wedge \beta) = M(\alpha) \cap M(\beta)$, and from A6 we get:

(T10) $\quad M(\sim \alpha) = SP - M(\alpha)$.

Thus $M(\alpha \vee \beta) = M(\alpha) \cup M(\beta)$; moreover, $M(\alpha \vee \sim \alpha) = SP$, $M(\alpha \wedge \sim \alpha) = \emptyset$.

The *objective* $S(\alpha)$ of a proposition α is the set of all its minimal verifiers: $S(\alpha) = Min(V(\alpha))$. Thus, in view of T1:

(T11) $\quad S(\alpha \wedge \beta) = S(\alpha) * S(\beta)$.

To determine now the objectives of tautology and negation observe firstly that both the family $\underline{M}$ of loci and the family $\underline{S}$ of objectives are subfamilies of $\underline{Min}$; moreover, $M(\alpha) \sim_v S(\alpha)$. Consider the partition $\underline{Min}/ \sim_v$. Its classes are sublattices of $\underline{Min}$, and the following holds good for them: for any $\alpha \in \underline{L}$,

T(12) $\quad S(\alpha) = inf|S(\alpha)|, M(\alpha) = sup|S(\alpha)|$,

the infima and suprema taken with regard to involvement. Hence for tautology and contradiction we have:

(T13) $\quad S(\alpha \vee \sim \alpha) = inf SP = \{\circ\}, S(\alpha \wedge \sim \alpha) = inf|\emptyset| = \{\lambda\}$.

Now for negation, expressing $S(\sim \alpha)$ as a function φ of $S(\alpha)$, we have:

(T14) $\quad S(\sim \alpha) = \varphi(S(\alpha)) = inf|SP - sup|S(\alpha)||$.

Similarly for disjunction we obtain by T9:

(T15) $\quad S(\alpha \vee \beta) = S(\alpha) \oplus S(\beta) = inf|S(\alpha) + S(\beta)|$.

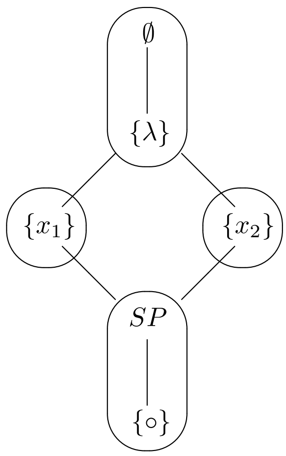
The map $inf|M|$ of $\underline{M}$ onto $\underline{S}$ is clearly one-to-one and a homomorphism with regard to the operations defined. Consequently, $\langle \underline{M}, \cap, \cup, -, SP, \emptyset \rangle$ and $\langle \underline{S}, *, \oplus, \varphi, \{\circ\}, \{\lambda\} \rangle$ are two isomorphic Boolean algebras, $\underline{M}$ being the representation of $\underline{S}$. Yet another is the family of complete sets of verifiers for $\underline{L}$, with $V(\alpha \vee \beta)$ being the least V-set containing both $V(\alpha)$ and $V(\beta)$, and $V(\sim \alpha)$ being the greatest V-set contained in $SE'' - (V(\alpha) - \{\lambda\})$; moreover, $SE'' = 1$, $\{\lambda\} = 0$. Hence either might be regarded as the ontological counterpart of $\underline{L}$. Possible-world semantics chooses $\underline{M}$, which recommends itself be its formal simplicity. $\underline{S}$ has other merits. One is the ease of defining a strong notion of independence for $\underline{L}$, crucial e.g. for the philosophy of logical atomism (cf. [3], [4]); i.e.,

$$Indep(\alpha, \beta) \text{ iff } \bigwedge_{x \in S(\alpha)} \bigwedge_{y \in S(\beta)} x; y \neq \lambda \wedge x!y = \circ.$$

It might be difficult to express this in terms of $\underline{M}$, at least without introducing a measure on it.

4. An Example

The simplest non-degenerate case of a logical space is a space of just one binary dimension: $SP = D = \{x_1, x_2\}$. This corresponds to a language $\underline{L}$ with only two atomic propositions, e.g. "switch on" and "switch off", representing the only two possible states of some definite device at some definite moment of time. The diagram below presents the corresponding lattice of minimal SE''-sets, with the dotted lines marking its V-equivalence classes.



References

[1] B. Wolniewicz, *On the Verifiers of Disjunction*, this**Bulletin**, vol. 9, no. 2 (1980), pp. 57–59.

[2] B. Wolniewicz, *On the Lattice of Elementary Situations*, this **Bulletin**, vol. 9, no. 3 (1980), pp. 115–121.

[3] B. Wolniewicz, *A Difference between Russell's and Wittgenstein's Logical Atomism*, **Proceedings of the XIVth International Congress of Philosophy**, Vienna 1968, vol. 2, pp. 263–267.

[4] B. Wolniewicz, *Four Notions of Independence*, **Theoria**, vol. 36 (1970), Part 2, pp. 161–164.

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