EVERY TWO-VALUED PROPOSITIONAL CALCULUS
HAS THE INTERPOLATION PROPERTY

It is known that two-valued calculi, one with implication, negation (+
other connectives) and the other pure implicational (cf. [1]) have the Inter-
polation Property. In this paper we prove that every two-valued calculus
with implication + other connectives has this property.

Let $L = (L, \text{Con})$ be an algebra of formulas formed in the usual manner
by means of propositional variables and operations from $\text{Con}$ denoted by
propositional connectives. We assume that $\text{Con}$ contains the implication
($\rightarrow$) and that all operations of $\text{Con}$ are finite. Let $M$ be a matrix (connected
with the language $L$) with the set $\{0, 1\}$ as the universum and 1 as the
distinguished value. We also assume that $\rightarrow$ is defined in $M$ in the usual
manner. Symbols $V_a$ ($a \in L$) and $T$ denote the set of variables of the
formula $a$ and the set of tautologies of the matrix $M$, respectively.

Theorem. If $a \rightarrow b \in T$ and $V_a \cap V_b \neq \emptyset$, then there exists a formula $c$
such that $V_c \subseteq V_a \cap V_b$ and $a \rightarrow c, c \rightarrow b \in T$.

Consider two cases: 1°. there exists an $n$ argument ($n \geq 0$) connective
$f$ of $\text{Con}$ such that $f(1, \ldots, 1) = 0$, 2°. such a connective does not exist.

If 1° holds, then the formula: $p \rightarrow f(p \rightarrow p, \ldots, p \rightarrow p)$ defines the
classical negation, and in this case the proof is well-known. Assume that the
second case holds. Denote the set of formulas $V_a \cap V_b \cup \{p \rightarrow p : p \in V_a \cap V_b\}$
by the symbol $W$.

Let $S$ be the set of all substitutions $s$ such that for every variable $p$ the
following conditions are satisfied:

if $p \in V_a - V_b$ then $sp \in W$,
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if \( p \not\in V_a - V_b \) then \( sp = p \).

Since the set \( V_a - V_b \) is finite, then the set \( S \) is also finite. So, let \( c = s_1 a \lor \ldots \lor s_k a \), where \( \{s_1, \ldots, s_k\} = S \). (The symbol \( \lor \) denotes the classical disjunction. This connective is obviously defined by \( \rightarrow \)). If \( s \in S \), then \( s(a \rightarrow b) = sa \rightarrow b \). Hence \( c \rightarrow b \in T \). Now, we prove that \( a \rightarrow c \in T \).

Suppose that for some valuation \( v \), \( va = 1 \) and \( vc = 0 \). Hence and by assumption 2\( \circ \), there is a variable \( p \in V_a \cap V_b \) such that \( vp = 0 \). Let \( s \) be a substitution such that for every variable \( q \):

\[
\begin{align*}
    \text{if } q \in V_a - V_b & \text{ then } sq = \begin{cases} 
        p & \text{if } vq = 0 \\
        p \rightarrow p & \text{otherwise}
    \end{cases} \\
    \text{if } q \not\in V_a - V_b & \text{ then } sq = q.
\end{align*}
\]

So for every variable \( q \), \( vq = vsq \). Hence \( va = vsa \). It can easily be noticed that \( s \in S \). So \( 1 = vsa \leq vc \). Contradiction.

References


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