A SIMPLER PROOF OF SAHLQVIST’S THEOREM ON COMPLETENESS OF MODAL LOGICS

This is a preliminary report of a part of the paper [6], not submitted for publication yet.

We assume familiarity with standard notions and notation in propositional modal logic (cf. e.g. [4] or [5]). As usual, we say that a (propositional normal modal) logic $L$ is complete if $\vdash_L P$ iff $|=L P$ (where $|=L P$ means that the formula $P$ is valid in all frames for $L$). Also, we say that a formula $P$ corresponds to a first-order formula $\chi_P$ if for each frame $F$, $F |= P$ iff $P$ satisfies the elementary property expressed by $\chi_P$. In [5], Sahlqvist proves the following:

**Theorem.** Let $S$ be any modal formula which is equivalent to a conjunction of formulas of the form $\Box^m (Q \rightarrow P)$ where

1. $P$ is positive,
2. after eliminating $\rightarrow$ from $Q$ and rewriting $Q$ with $\neg$ occurring only in front of propositional variables, no positive occurrence of a variable is in a subformula of $Q$ of the form $P_1 \lor P_2$ or $\Diamond P_1$ within the scope of some $\Box$.

Then $KS$ (= the logic obtained from $K$ adding $S$ as an axiom schema) is complete and $S$ corresponds to a first-order formula $\chi_S$ effectively obtainable from $S$.

We believe that the range of applications of this theorem is so wide (e.g. each of $D, T, B, S4, K4, S5, \ldots$ is included) that the effort for simplifying its proof is worthwhile. Our simplification is based on the introduction of a topology in each first-order frame, as done in [6] to which we refer for more
details. Any first-order frame $F = (X, R, T)$ ($T$ is a boolean subalgebra of $2^X$ which is closed under an operation $R^*$ defined by $R^* C = \{x \in X : xRy \Rightarrow y \in C\}$, for each $C \subseteq X$) is topologized taking $T$ as a base for open subsets (and hence also for closed subsets, since each element of $T$ becomes open and closed). Then we say that $(X, R, T)$ is descriptive if the topology induced by $T$ is compact and Hausdorff and if $R^n x = \{y : x R^n y\}$ is closed for each $n$ and each $x \in X$ (the equivalence of this definition to that in [2] is almost obvious). The usefulness of this notion rests on two facts:

a) each first-order frame is equivalence to a descriptive frame;

b) all first-order canonical frames are descriptive.

The main method to prove completeness of a logic $L$ is to show that it is canonical, i.e. that $L$ holds on its canonical frame (as a matter of fact, by this method one obtains strong completeness, i.e. completeness for the deductive system $\vdash_L$). Combining this with b) and recalling that each logic holds on its first-order canonical frame, to show completeness of $KS$ it is enough to show that $S$ is persistent, i.e. that for each descriptive $F = (X, R, T)$, $(X, R, T) \models S$ implies $(X, R) \models S$.

Now let us say that a formula is plain if it is obtained starting from formulas of the form $\Box m p_i$ and negative formulas by applying only $\land$ and $\Diamond$. Then, with no difficulty but with some patience, one can prove that any formula $Q$ satisfying the condition 2. of the theorem is equivalent to a disjunction of plain formulas. So the formula $S$ of the theorem may be rewritten as a conjunction of formulas of the form $\Box^m (Q \rightarrow P)$, where $Q$ is plain and $P$ is positive. Therefore the proof of the theorem reduces to a proof of:

**Lemma 1.** If $Q$ is a plain formula, $P$ a positive formula and $m \geq 0$, then $S = \Box^m (Q \rightarrow P)$ is persistent and corresponds to a first-order formula $\chi_S$.

The proof for correspondence is quite similar to that in [5], but we have to repeat it here since persistence is obtained on the way. Spelling out definitions, one easily finds out that with each modal formula $P(p_1, \ldots, p_k)$ one can associate a formula $c_P(u, S_1, \ldots, S_k)$, which is first-order except for unary predicate variables $S_i$ and in which $u$ is the only free individual variable, such that for any frame $F = (X, R, T)$ and any $x \in X$, $F \models P$ iff $(\forall S_1 \ldots S_k \in T) c_P(u, S_1, \ldots, S_k)$ holds in $(X, R)$ at $x$ (cf. [3], pp. 34–35).
Using the abbreviations: $x \in S_i$ for $S_i(x)$, $R^m x \subseteq S_i$ for $x R^m y \to y \in S_i$ and \( \bigcup_{j \leq k} R^m x_j \subseteq S_i \) for $R^m x_1 \subseteq S_i \land \ldots \land R^m x_k \subseteq S_i$, one can show by induction that when $Q$ is plain, $c_Q(u)$ may be written in the form:

\[ (*) \exists z_1 \ldots \exists z_n [A \land \bigcup_{j \leq m_1} R^{n+1} y_j \subseteq S_1 \land \ldots \land \bigcup_{j \leq m_k} R^{n+1} y_j \subseteq S_k \land c_{N_1}(y_{h_1}, S_1, \ldots, S_k) \land \ldots c_{N_i}(y_{h_i}, S_1, \ldots, S_k)] \]

where $A$ is a conjunction of atomic formulas of the form $y_i R y_j$; all $y_i$'s are among $z_1, \ldots, z_n$ and $u$, $N_1, \ldots, N_i$ are negative formulas and all indices may be equal to zero.

Hint. If $Q = \square^m p_i$, then $c_Q$ is $R^m u \subseteq S_i$, and if $Q$ is negative the claim is trivial. Inductive cases (only $\land$ and $\square!$) are mere manipulations. To get a grasp of what is going on, we suggest the reader to try on an easy formula.

Now let $S = \sqcap^m (Q \to P)$ where $Q$ is plain and $P$ is positive. Then $c_S(u) = \forall w (u R^m w \to (c_Q(w) \to c_P(w)))$. Using (*) where we write $U_i$ for \( \bigcup_{j \leq m_i} R^{n+1} y_j \) we can write $c_S$ as

\[ \forall u \forall z_1 \ldots z_n (u R^m w \land A \to (\bigwedge_{j \leq k} (U_i \subseteq S_i) \to (\bigvee_{i \leq l} c_{N_i'}(y_{h_i}) \lor c_P(w)))) \]

where each $N_i'$ is positive. For convenience, let $A' = u R^m w \land A$ and $B(S_1, \ldots, S_k) = \bigvee_{i \leq l} c_{N_i'}(y_{h_i}) \lor c_P(w)$. So for any first-order frame $F = (X, R, T)$, we have that: $F \models_x S$ iff

\[ (**) \forall S_1 \ldots S_k \in T) \forall w \forall z_1 \ldots z_n (A' \to (\bigwedge_{i \leq k} (U_i \subseteq S_i) \to B(S_1, \ldots, S_k))) \]

holds at $x$.

To prove correspondence, suppose $(X, R) \models_x S$. Then evaluating $S_1, \ldots, S_k$ in $(**)$ on $U_1, \ldots, U_k$, we have that

\[ \chi_S : \forall w \forall z_1 \ldots z_n (A' \to B(U_1, \ldots, U_k)) \]

holds at $x$. Note that $\chi_S$ is a first-order formula.

Conversely, suppose $\chi_S$ holds at $x$ and suppose $A'$ holds for some given values of $w, z_1, \ldots, z_n$. Then for such values $B(U_1, \ldots, U_k)$ holds and hence, since $B$ comes from a positive modal formula, also $B(S_1, \ldots, S_k)$ holds whenever $U_i \subseteq S_i$. So we have proved that
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\((**)'\) \((\forall S_1 \ldots S_k \subseteq X) \forall w \forall z_1 \ldots z_n (A' \rightarrow ( \bigwedge_{i \leq k} (U_i \subseteq S_i) \rightarrow B(S_1, \ldots, S_k)))\)

holds at \(x\). But \((**)'\) is nothing else than \((**)\) written for the frame \((X, R)\), and therefore \((X, R) \models x S\).

To prove persistence, it is enough to show that on any descriptive \(F = (X, R, T)\), \((**)\) implies \(\chi_S\). For in this case also \((**)'\) holds by the above correspondence of \(S\) to \(\chi_S\) and hence also \((X, R) \models x S\).

The following fact is the key of our simplification (cf. also \[2\], section 14):

**Lemma 2.** Let \(P\) be a positive formula, \((X, R, T)\) a descriptive frame and \(Y\) any subset of \(X\). Then

\[\bigcap_{Y \subseteq C \in T} \{x \in X : c_P(., C, .) \text{ holds at } x\} = \{x \in X : c_P(., \overline{Y}, .) \text{ holds at } x\},\]

where \(\overline{Y}\) is the (topological) closure of \(Y\).

**Proof.** By induction on the structure of \(P\) (first write \(P\) without \(\neg\)). The step for \(\lozenge\) is substantially the following fact (cf. \[1\], Lemma 3; compactness of the frame is essential here): for any filter \(H\) of closed subsets of \(X, R^{-1}(\bigcap H) = \bigcap_{C \in H} R^{-1}C\).

Now rewrite \((**)\) as

\[\forall w \forall z_1 \ldots z_n (A' \rightarrow (\forall S_1 \ldots S_k \in T)(\forall i \leq k (U_i \subseteq S_i) \rightarrow B(S_1, \ldots, S_k))).\]

Then by Lemma 2, the subformula to the right of \(A'\) holds if \(B(U_1, \ldots, U_k)\) holds. By the definition of descriptive frames, each \(U_i\) is a closed subset of \(X\), hence \(U_i = \overline{U_i}\) and \(\chi_S\) holds.

As a final comment, note that we have proved something more than required: correspondence and persistence of \(S\) are in fact proved to hold at each point (following \[7\], we would say that they hold locally).

**References**


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