ON INTERMEDIATE LOGICS WHICH CAN BE
AXIOMATIZED BY MEANS OF IMPLICATIONLESS
FORMULAS

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It has been proved by McKay [3] that an implicationless formula is a
theorem of the intuitionistic propositional logic (INT) if and only if it is
in the content of every pseudo-Boolean algebra of the form $\mathfrak{B}_{\oplus}$ where $\mathfrak{B}$
is a finite Boolean algebra. In this paper we shall give a generalization of
the above criterion of McKay and as a corollary we shall obtain a complete
description of all intermediate logics which can be axiomatized by means
of implicationless formulas. These results are closely related to those of
Lee [5] who gave a description of all varieties of distributive lattices with
pseudocomplementations.

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his help during the research.

We shall use a common set-theoretical and algebraic notation (see for
example [1]). In particular by German capitals $\mathfrak{A}, \mathfrak{B}, \ldots$ we denote algebras
and by corresponding Latin capitals $A, B, \ldots$ their domains.

Let $\mathfrak{F} = \langle F, \land, \lor, \to, \neg \rangle$ be the absolutely free algebra of formulas of
type $(2, 2, 2, 1)$ free generated by a countably infinite set of variables $V$. Let
$\mathfrak{F}_0 = \langle F_0, \land, \lor, \neg \rangle$ be the subalgebra of the $\{\land, \lor, \neg\}$ – reduct of $\mathfrak{F}$ generated
by the set $V$. Then $\mathfrak{F}_0$ is the absolutely free algebra of type $(2, 2, 1)$ and
$F_0$ is the set of all implicationless formulas. The class of pseudo-Boolean
algebras will be denoted by $\triangle$ and the class of all distributive lattices with
pseudocomplementations by $\triangle_0$. For $\mathfrak{A} \in \triangle$ we put $E(\mathfrak{A}) = \{ \alpha \in F | v(\alpha) = 1 \}$ for all $v \in Hom(\mathfrak{F}, \mathfrak{A})$ and
$E_0(\mathfrak{A}) = E(\mathfrak{A}) \cap F_0$. For any $\mathfrak{A}, \mathfrak{D} \in \triangle \setminus (\triangle_0)$
such that $A \cap B = \{1_A\} = \{0_D\}$ the symbol $A \oplus D$ will denote the sum of $A$ and $D$ in the sense of Troelstra [6]. The symbol $2$ will be reserved for the two-element Boolean algebra. We shall write $A \oplus$ instead of $A \oplus 2$.

The symbol $Rg(A)$ denotes the Boolean algebra of regular elements of the algebra $A$.

First let us note the following key Lemma:

**Lemma.** Let $A, D \in \Delta_0$ be such that $A \cap B = \{1_A\} = \{0_D\}$. Let $\Theta_1$ be a congruence relation of $A$ such that $[0_A]_{\Theta_1} = \{0_A\}$ and let $\Theta_2$ be an arbitrary congruence relation of $D$. Then there exists a congruence relation $\Theta$ of the algebra $A \oplus D$ such that $\Theta_1 = \Theta \cap (A \times A)$ and $\Theta_2 = \Theta \cap (B \times B)$.

Applying the above lemma one can easily prove the following proposition which is a generalization of the criterion of McKay [3]:

**Proposition.** For every $A \in \Delta$

$$E_0(A \oplus) = E_0(Rg(A) \oplus)$$

We put $K_n = \{A \in \Delta \mid |Rg(A)| \leq 2^n\}$, $n = 0, 1, 2, \ldots$, and $K_\omega = \{A \in \Delta \mid |Rg(A)| < \aleph_0\} = \bigcup_{n=0}^\infty K_n$. We define a sequence of intermediate logics: $L_n = \bigcap \{E(A \oplus) \mid A \in K_n\}$, $n = 0, 1, 2, \ldots$, and $L_\omega = \bigcap_{n=0}^\infty L_n = \bigcap \{E(A \oplus) \mid A \in K_\omega\} = INT$.

The following theorems give a complete characterization of intermediate logics axiomatizable by means of sets of implicationless axioms:

**Theorem 1.** For every $X \subseteq F_0$ the intermediate logic $C(X)$ (where $C$ is the consequence operation in $\mathfrak{A}$ determined by $INT$ and the rules of substitution and detachment) is identical with one of the logics $L_n$ ($n = 0, 1, 2, \ldots, \omega$).

**Theorem 2.** $L_n = C(\alpha_n)$, $n = 0, 1, 2, \ldots, \omega$, where

- $\alpha_0 = x \lor \neg x$,
- $\alpha_1 = \neg x \lor \neg \neg x$,
- $\alpha_2 = \neg (x_1 \land x_2) \lor \neg (\neg x_1 \land x_2) \lor \neg (x_1 \land \neg x_2)$,
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\[ \alpha_n = \neg(x_1 \land \ldots \land x_n) \lor \neg(\neg x_1 \land x_2 \land \ldots \land x_n) \lor \ldots \lor \neg(x_1 \land \ldots \land x_{n-1} \land \neg x_n), \]

\[ \ldots \]

\[ \alpha_\omega = \neg(x \land \neg x). \]

It is easy to see that each of the logics \( L_n \) can be axiomatized by means of a disjunctionless formula, so in view of the well known criterion of McKay [4] each \( L_n \) is finitely approximable and thus decidable by virtue of Harrop [2].

References