This is an abstract of part of a planned book on the problem of verisimilitude.

A metric operation \(d\) is one that satisfies the three postulates

1. \(d(a, c) \leq d(a, b) + d(c, b)\)
2. \(d(a, c) = 0 \rightarrow a = c\)
3. \(d(a, a) = 0\).

We present here postulates, written in terms of a metric operation \(d\), that are necessary and sufficient for a finite lattice \(L\) to be (1) modular, (2) distributive, and (3) Boolean. Work on these problems was started in [4] and [2], and summarized in [3].

1. A metric operation \(d\) that is defined for any two comparable elements of a lattice \(L\) will be called here a chain metric.

**Theorem 1.** \(L\) is modular if and only if there may be defined on its comparable pairs a chain metric \(d\) that satisfies the postulate

\[
(4) \quad d(a, c, a) = d(c, a + c),
\]

together with one or more of

\[
(5) \quad c < b < a \rightarrow d(a, b) < d(a, c) \\
(6) \quad c < b < a \rightarrow d(b, c) < d(a, c) \\
(7) \quad c \leq b \leq a \rightarrow d(a, b) + d(c, b) = d(a, c).
\]
Theorem 2. Let $d$ be a chain metric on $L$. Then if $d$ satisfies both (4) and (7) it may be extended to a metric on the whole of $L$ that satisfies

(8) $d(a.c, a + c) = d(a, c)$.

Theorem 2 does not hold when (7) is replaced by (5) or (6), or by both of them together.

Theorem 3. Let $d$ be a metric on $L$ for which (7) and

(9) $d(a.c, a + c) \leq d(a, c)$

hold. Then both (4) and (8) hold.

Theorem 4. $L$ is modular if and only if there may be defined on it a metric operation $d$ for which (7) and (9) both hold.

The effect of Theorem 3 and 4 is that if $L$ is modular then there is a non-trivial metric $d$ for which the lattice quadrangles of $L$ are all rectangles.

Theorem 5. $L$ is modular if and only if there may be defined on it a metric operation $d$ for which

(10) $d(a.b, c.b) + d(a + b, c + b) \leq d(a, c)$

holds.

Formula (1) is shown in [1], pp. 76f. to hold in every so-called metric lattice. (A metric lattice is a modular lattice in which a distance function is defined in terms of a positive valuation). It can be shown to be equivalent to the conjunction of (7) and (9).

2. Distributive lattices can be characterized in ways closely similar to Theorem 1 and 5.

Theorem 6. $L$ is distributive if and only if there may be defined on its comparable pairs a chain metric $d$ that satisfies (4), together with one or more of (5), (6), (7), and, in addition,

(11) $d(a.c, a) \neq d(a.c, c)$, if $a \neq c$. 
It follows that if $L$ is distributive then there is a non-trivial metric $d$ for which the lattice quadrangles are all rectangles, but none of them is a square.

**Theorem 7.** $L$ is distributive if and only if there may be defined on it a metric operation $d$ for which

\[ (12) \quad d(a,b,c,b) + d(a + b, c + b) = d(a,c) \]

holds.

A slightly different characterization of distributive lattices is given as follows.

**Theorem 8.** $L$ is distributive if and only if there may be defined on it a metric operation $d$ for which

\[ (13) \quad d(a, b) + d(c, b) = d(a.c, b) + d(a + c, b). \]

That is, the function $v(a) = d(a, b)$ is for any $b$ a valuation (though not necessarily a positive valuation).

3. A lattice $L$ is said to have complements if for each $a$ in $L$ there is a $c$ in $L$ such that

\[ (14) \quad a.c \leq b \leq a + c \]

for every $b$ in $L$. It is well known that a modular lattice with complements is distributive if an only if the complements are unique. In this case it is called a Boolean lattice.

**Theorem 9.** $L$ is Boolean if and only if there may be defined on it a metric operation $d$ for which

\[ (15) \quad \exists \forall b [d(a,b) + d(c,b) = d(a,c)], \]

together with $(11)$, $(12)$, or $(13)$, holds.

In the presence of $(15)$ we can drop $(3)$; and, if we include $(13)$, we can also drop the absorption identities

\[ (16) \quad a.(a + b) = a = a + a.b \]

from the definition of a lattice. The remaining clauses (that $\cdot$ and $+$ are
commutative and associative) are not easily replaced by metric postulates, though the extremely artificial formula

\[ d(a, (c.b), (b.a), c) = 0 = d(a + (c + b), (b + a) + c) \]

would suffice.

**Theorem 10.** Let \( L \) be any finite set on which are defined commutative and associative operations \( . \) and \( + . \) Then \( L \) is a Boolean lattice if an only if there may be defined on it a function \( d \) for which

1. \( d(a, c) \leq d(a, b) + d(c, b) \)
2. \( d(a, c) = 0 \iff a = c \)
13. \( d(a, b) + d(c, b) = d(a.c, b) + d(a + c, b) \)
15. \( \exists c \forall b[(a, b) + d(c, b) = d(a, c)] \)

all hold.

If a (unique) complementation operation \( − \) is introduced into a Boolean lattice \( L \) then \( L \) becomes a Boolean algebra. We may easily adapt postulate (15) to obtain the following metric characterization of Boolean algebras.

**Theorem 11.** Let \( L \) be a lattice with an operation \( − \). Then \( L \) is a Boolean algebra (and \( − \) its complementation operation) if and only if there may be defined on it a function \( d \) for which (1), (2), (10) and

18. \( d(a, b) + d(−a, b) = d(a, −a) \)

hold.

It can be seen that postulates (1), (10), and (18) are very similar to the axioms \((\Pi), (\varrho), (\sigma)\) given on p. 62 of [5]; indeed, these latter axioms are sufficient to guarantee that \( L \) is a Boolean algebra.

4. All theorems stated above may be generalized from finite to arbitrary lattices if we allow the metric operation \( d \) to take values not only in the real numbers, but in some ordered extension field of the real numbers. In addition, postulate (2) may be dropped provided that in each case we factor the lattice \( L \) by the congruence relation \( d(a, c) = 0 \). There are a number of further generalizations, which I hope to recount elsewhere.
References


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