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A SEMANTICAL STUDY OF SOME SYSTEMS OF VAGUENESS LOGIC

1. Introduction

In [1] we have characterized four types of vagueness related to negation, and constructed the corresponding propositional calculi adequate to formalize each type of vagueness. The calculi obtained were named $\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2$ and $\mathcal{C}_1$ ($\mathcal{C}_1$ is the first system of the hierarchy of paraconsistent logic of da Costa [2]). The relations among these calculi and the classical propositional calculus $\mathcal{C}_0$ can be represented in the following diagram, where the arrows indicate that a system is a proper subsystem of the other.

\[
\begin{array}{ccc}
\mathcal{V}_0 & \xrightarrow{\text{proper}} & \mathcal{V}_1 \\
& \xrightarrow{\text{proper}} & \mathcal{V}_2 \\
& & \xrightarrow{\text{proper}} \mathcal{C}_1 \\
& & \xrightarrow{\text{proper}} \mathcal{C}_0
\end{array}
\]

In this paper we present a two-valued semantics for each of these systems. The semantics used here is the Henkin-style semantics which has proven fruitful in treating other paraconsistent logics (see, for example, [3]).

The terminology and notations here are those of [1]. The postulates of $\mathcal{V}_0$ are the following (with the definitions: $\mathcal{V}_0 A =_{df} A \lor \neg A, \mathcal{V}_1 A =_{df} \neg (A \land \neg A), \mathcal{V}_1 A =_{df} A \lor (\neg A \land A)$):
1. $A \supset (B \supset A)$
2. $(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$
3. $A, A \supset B/B$
4. $A \supset (B \supset A \& B)$
5. $A \& B \supset A$
6. $A \& B \supset B$
7. $A \supset A \lor B$
8. $B \supset A \lor B$
9. $(A \supset C) \supset ((B \supset C) \supset (A \lor B \supset C))$
10. $^0A \& B^0 \& (A \supset B) \& (A \supset \neg B) \supset \neg A$
11. $A^0 \& B^0 \supset (A \supset B)^0 \& (A \& B)^0 \& (A \lor B)^0$
12. $^0A \& ^0B \supset (A \supset B) \& ^0(A \& B) \& ^0(A \lor B)$
13. $A^0 \supset (\neg A)^0$
14. $^0A \supset (^0 \neg A)$
15. $\neg^* \neg^* A \supset A$

The postulates of $V_1$ are 1-14 of $V_0$ plus:

15. $\neg^* \neg^* A \supset A$
16. $A \lor \neg^* A$

where $\neg^* A$ is defined by $A^0 \& (A \supset \neg A)$.

The postulates of $V_2$ are 1-9 of $V_1$ plus:

10. $^0A \& (A \supset B) \& (A \supset \neg B) \supset \neg A$
11. $^0A \& ^0B \supset ^0 (A \supset B) \& ^0(A \& B) \& ^0(A \lor B)$
12. $^0A \supset ^0 (\neg A)$
13. $\neg (A \& \neg A)$
14. $\neg^* \neg^* A \supset A$

where $\neg^* A$ is defined by $A \supset \neg A$.

2. The semantics of $V_0$

Let $F$ denote the set of formulas of $V_0$. $\Gamma$ will designate any subset of $F$. The set $\{A \in F : \Gamma \vdash A\}$ will be denoted by $\Gamma$.

Definition. The set $\Gamma$ of formulas is said to be trivial in $V_0$ if $F = \Gamma$; otherwise, $\Gamma$ is called nontrivial.
Definition. \( \Gamma \) is a maximal nontrivial set if it is nontrivial and, for all \( A \), if \( A \not\in \Gamma \), then \( \Gamma \cup \{A\} \) is trivial.

Theorem 1. If \( \Gamma \) is maximal nontrivial, then:
\[
\begin{align*}
\Gamma \vdash A & \iff A \in \Gamma \\
\neg A \in \Gamma & \implies \neg A \not\in \Gamma \\
A \supset B & \in \Gamma \implies B \in \Gamma \\
A & \in \Gamma \implies \neg \neg A \not\in \Gamma \\
A \not\in \Gamma & \implies \neg A \in \Gamma
\end{align*}
\]

Definition. A valuation for \( V \) is a function \( \vartheta : F \to \{0, 1\} \) such that:
\[
\begin{align*}
1. \quad \vartheta(A \supset B) &= 1 \iff \vartheta(A) = 0 \text{ or } \vartheta(B) = 1 \\
2. \quad \vartheta(A \& B) &= 1 \iff \vartheta(A) = \vartheta(B) = 1 \\
3. \quad \vartheta(A \lor B) &= 1 \iff \vartheta(A) = 1 \text{ or } \vartheta(B) = 1; \\
4. \quad \vartheta(A^0) &= \vartheta(B^0) = 1 \implies \vartheta((A \supset B)^0) = \vartheta((A \& B)^0) = \vartheta((A \lor B)^0) = 1; \\
5. \quad \vartheta(A^0) &= \vartheta(B^0) = 1 \implies \vartheta((A \lor B)^0) = \vartheta((A \& B)^0) = \vartheta((A \supset B)^0) = 1; \\
6. \quad \vartheta(A^0) &= 1 \implies \vartheta(\neg(A^0)) = 1 \\
7. \quad \vartheta(0^A) &= 1 \implies \vartheta(0^A(\neg A)) = 1; \\
8. \quad \vartheta(A) &= \vartheta(\neg A) = 1 \implies \vartheta(A^0) = 0.
\end{align*}
\]

Lemma. If \( \vartheta \) is a valuation for \( V \) then \( \vartheta(A) = 1 \iff \vartheta(\neg A) = 0 \).

Definition. The formula \( A \) is valid in \( V \) if for each valuation \( \vartheta \), \( \vartheta(A) = 1 \).

A valuation \( \vartheta \) is a model of the set \( \Gamma \) if \( \vartheta(A) = 1 \) for all formulas \( A \in \Gamma \).

If each model of \( \Gamma \) is a model of \( \{A\} \) we write \( \Gamma \models A \). (In particular, \( \models A \) means that \( A \) is valid.)

Theorem 2 (Theorem of Soundness). \( \Gamma \vdash A \implies \Gamma \models A \).

Proof. By induction on the length of a deduction of \( A \) from \( \Gamma \).

Lemma. If \( \Gamma \) is nontrivial, then \( \Gamma \) is contained in a maximal nontrivial set.

Proof. By an obvious adaptation of the corresponding classical theorem.

Lemma. Every maximal nontrivial set of formulas has a model.

Proof. We define the function \( \vartheta : F \to \{0, 1\} \) as follows: for every formula \( A \), if \( A \in \Gamma \), then \( \vartheta(A) = 1 \), otherwise \( \vartheta(A) = 0 \). Then we prove that \( \vartheta \) is a valuation for \( V \).

Theorem 3 (Theorem of Completeness). \( \Gamma \models A \implies \Gamma \vdash A \).
Proof. Consequence of the preceding lemmas.

As an application of the semantics we can prove the following theorem.

**Theorem 4.** In $V_0$, the following schemas (among others) are not valid:

- $(A \lor \neg A) \lor \neg (A \land \neg A)$
- $A \lor \neg A$  
- $\neg (A \land \neg A)$
- $A \lor A^0$  
- $\neg A \lor A^0$  
- $A \supset (\neg \neg A \supset A)$
- $0A \lor (A \land \neg A)$
- $A \lor A^0$  
- $\neg (A \land \neg A)$
- $0A \land A^0$.

3. **The semantics of the system $V_1$**

A semantics for the system $V_1$ can be obtained by strengthening clause 8 of Definition 3, in the following way:

- $8’. \vartheta(A) = \vartheta(\neg A) = 1 \iff \vartheta(A^0) = 0$.

With this definition of valuation it is easy to prove the theorems of soundness and completeness. We need only to show that function $\vartheta$ which appears in the proof of the last lemma satisfies the clause $8’$ of the definition of valuation for $V_1$, and this is not difficult.

As an application of the semantics we can prove the following result:

**Theorem 5.** In $V_1$, the following schemas (among others) are not valid:

- $A \supset \neg \neg A$, $\neg \neg A \supset A$, $A \lor \neg A$, $\neg (A \land \neg A)$.

**Theorem 6.** $V_2$ is a proper subsystem of $V_1$.

Proof. It is enough to verify that axiom 15 of $V_0$ is a theorem of $V_1$, and that there are formulas which are not valid in $V_0$ are theorems of $V_1$ (for instance, $0A \lor A^0$).

4. **The semantics of the system $V_2$**

$V_2$ is obtained from $V_1$ by adding as new axiom the law of contradiction, $\neg (A \land \neg A)$. A semantics for the system $V_2$ can be obtained by adding in the definition of valuation for $V_1$, the following clause:

- $9. \vartheta(A) = 1 \Rightarrow \vartheta(\neg A) = 0$.

With this definition of valuation, it is easy to prove the theorems of soundness and completeness. We will only show that the new axiom is valid in $V_2$. Suppose not; then there is a valuation $\vartheta$, such that $\vartheta(\neg (A \land \neg A)) =$
0. Then, by clause 8’, we have \( \vartheta(A) = \vartheta(\neg A) = 1 \), which contradicts clause 9.

As an application of the semantics, we have the following result:

Theorem 7. In \( V_2 \), the following schemas (among others) are not valid:

\[ \neg\neg A \supset A, \ A \lor \neg A, \ \neg A \lor \neg B \supset \neg(\neg A \& B), \ (A \supset B) \supset \neg A \lor B. \]

Theorem 8. \( V_1 \) is a proper subsystem of \( V_2 \).

Proof. It is immediate, taking into account the construction of \( V_2 \) and the fact that \( \neg(A \& \neg A) \) is not a theorem of \( V_1 \) (see Theorem 5).

5. Final remarks

Observe that the semantics of \( V_1 \) can also be extended in order to obtain a semantics for the system \( C_1 \) of da Costa. This can be done by adding the following clause in the definition of valuation:

\[ 9'. \ \vartheta(A) = 0 \Rightarrow \vartheta(\neg A) = 1. \]

It is easy to show that this definition of valuation is equivalent to that formulated, for example, in [3].

Finally, we observe that a decision method for each of the system \( V_0 \), \( V_1 \), and \( V_2 \) can be obtained by an adaptation of the method proposed in da Costa and Alves [3].

References


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