A NOTE ON INCOMPLETENESS OF MODAL LOGICS WITH RESPECT TO NEIGHBOURHOOD SEMANTICS

This is a summary of a lecture read at the Seminar of the Department of Mathematical Logic held by Professor Jerzy Kotas, Institute of Mathematics, N. Copernicus University, Toruń, June 1978.

§0. By a modal logic we understand a proper subset of the set of propositional modal formulae that contains all classical tautologies, the axiom $\square(p \rightarrow q) \rightarrow (\square p \rightarrow \square q)$ and closed under modus ponens, substitution and necessitation. In our considerations all neighbourhood frames are normal, i.e. such that neighbourhoods of each point constitute a filter.

For a neighbourhood frame $F = (U, N)$, by $F^+$ we denote the algebra $(P(U), \cup, \cap, \neg, \square_N, 0, 1)$; where $(P(U), \cup, \cap, \neg, 0, 1)$ is the well known Boolean algebra and $\square_N$ is a unary operation defined as follows: $\square_N(S) = \{x \in U | S \in N(x)\}$ for every $S \subseteq U$. We know that $E(F) = E(F^+)$, where $E(F)$ ($E(F^+)$) is the set of all formulae which are valid in $F$ ($F^+$). Following Fine [3], for a modal logic $L$, we put $\delta^*(L) = \text{card}\{L' | L' \text{ is a modal logic such that for every neighbourhood frame } F, L' \subseteq E(F) \text{ iff } L \subseteq E(F)\}$. Our aim is to prove the following theorem which is a counterpart of Blok’s one (see [1], also [2]).

**Theorem 1.** For any modal logic $L$:

1) if $\square p \rightarrow \Diamond p \in L$ and $\square^np \rightarrow \square^{n+1}p \in L$ for some $n \geq 0$, then $\delta^*(L) = 2^{\aleph_0}$
2) if $p \rightarrow p \in L$, then $\delta^*(L) = 2^{\aleph_0}$.

§1. In order to prove the first part of Theorem 1 let us take into consideration the formulae
\( \alpha_{n,k} = (p \land \Diamond^{2^n}q) \rightarrow (\Diamond^{n}q \lor \Diamond^{2^n}(q \land \Diamond^{k(n+1)p})) ; \ n \geq 1, k \geq 1 \)

\( \beta_n = (\Box^n p \land \sim \Box^{n+1}p \land \sim \Box^{2n+1}p) \rightarrow \Diamond^n (\Box^{2n+1}p \land \sim \Box^{2n+2}p \land \sim \Box^{3n+2}p), \ n \geq 1 \)

\( \gamma_n = (\Box^n p \land \sim \Box^{n+1}p \land \sim \Box^{2n+1}p) \rightarrow \sim (\bigwedge_{1 \leq i \leq 2n+4} \Box^n (r \rightarrow \Diamond^n q_i) \land \bigwedge_{1 \leq i \neq j \leq 2n+4} \Box^n \sim (q_i \land q_j)), \ n \geq 1. \)

These formulae can be found in Thomason [5].

**Lemma 1.** For any \( n \geq 1 \) and every neighbourhood frame \( F \):

if \( \alpha_{n,k}(k \geq 1)\beta_n, \gamma_n \in E(F) \), then

\( \Box^n p \rightarrow (\Box^{n+1}p \lor \Box^{2n+1}p) \in E(F). \)

In the proof the ideas from Gerson [4] are used. Let us recall Blok’s definition of a family of modal algebras \( A_m \) (cf. [1]). \( A_m \) is a modal algebra of finite and cofinite subsets of the set of natural numbers \( N \). The operation \( \Box_m \) in \( A_m \) corresponding to a connective \( \Box \) is defined as follows:

\[
\Box_m = \begin{cases} 
\emptyset, & \text{if } M \text{ is finite} \\
[m_{i+1}, \infty), & \text{if } N \neq M \text{ is cofinite and } i = \min\{j|[m_j, \infty) \subseteq M\} \\
N, & \text{if } M = N
\end{cases}
\]

where \( m = (m_i)_{i=1}^{\infty} \) is a sequence of natural numbers satisfying \( m_1 = 3, m_2 = 4, m_{i+1} > m_i \) and \( m_{i+1} - m_i \leq 2 \), for \( i \geq 1 \).

**Lemma 2.** For each algebra \( A_m \)

\( \alpha_{n,k}, \beta_n, \gamma_n \in E(A_m) \) (\( n \geq 1, k \geq 1 \)).

For any class \( K \) of algebras, \( V(K) \) denotes the smallest variety that contains \( K \), and \( V(K)_{SI} \) is the class of all subdirectly irreducible members of \( V(K) \). The next lemma is an immediate consequence of Theorem 4.4 in Blok [1].

**Lemma 3.** For every algebra \( A_m \) and natural number \( n \geq 1 \): if \( B \in V(A_m)_{SI} \) and \( \Box^n p \rightarrow (\Box^{n+1}p \lor \Box^{2n+1}p) \in E(B) \), then \( B \cong 2 \).
Lemma 4. For a neighbourhood frame \( F \). Indeed, for them we have (cf. Lemma 1).

\[ \gamma = (\Box p \land \sim p) \rightarrow \Box^2(\Box^2 p \land \sim \Box p) \]

\[ \alpha = (p \land \Box^4 q) \rightarrow (\Box^{n+1} r \rightarrow \Box^n r) \lor \Box^2 q \lor \Box^4 (q \land \Box^4 p) \]

\[ n \geq 0, k \geq 3 \]

\[ \beta = (\Box p \land \sim p) \rightarrow \Box^2(\Box^2 p \land \sim \Box p) \]

\[ \gamma = (\Box p \land \sim p) \rightarrow (r \land \bigwedge_{1 \leq i \leq 5} \Box^2(r \rightarrow \Box^2 q_i) \land \bigwedge_{1 \leq i \leq 5} \Box^2(q_i \land r) \land \bigwedge_{1 \leq i \neq j \leq 5} (q_i \land q_j)) \]

These formulae will play a similar role to that in the previous section. Indeed, for them we have (cf. Lemma 1).

Corollary 1. Let \( L \) be a modal logic such that \( \Box p \rightarrow \Diamond p \in L \) and for some \( n \geq 1 \) the formulas \( \alpha_n \) (\( k \geq 1 \)), \( \beta_n \) and \( \gamma_n \) are theses of \( L \). Then, for every algebra \( A_m \) and neighbourhood frame \( F \), \( F^+ \in V(K_L \cup \{A_m\}) \) iff \( F^+ \in K_L \).

\[ \text{Let us suppose } \Box p \rightarrow \Diamond p \in L \text{ and } \Box^n p \rightarrow \Box^{n+1} p \in L, \text{ for some } n \geq 0. \]

Blok [1] has proved that \( V(K_L \cup \{A_m\}) \neq V(K_L \cup \{A_m\}) \), for every \( m \neq n \). But \( L \) contains also the formulæ \( \alpha_{n+1, k} \) (\( k \geq 1 \)), \( \beta_{n+1} \), and \( \gamma_{n+1} \), and so, by Corollary 1, we receive \( \delta(L) = 2^\aleph_0 \).

§2. Now, similarly, we prove the second part of Theorem 1. Therefore take the following formulae:

\[ \alpha_{n,k} = (p \land \Box^4 q) \rightarrow ((\Box^{n+1} r \rightarrow \Box^n r) \lor \Box^2 q \lor \Box^4 (q \land \Box^4 p)) \]

\[ n \geq 0, k \geq 3 \]

\[ \beta = (\Box p \land \sim p) \rightarrow \Box^2(\Box^2 p \land \sim \Box p) \]

\[ \gamma = (\Box p \land \sim p) \rightarrow (r \land \bigwedge_{1 \leq i \leq 5} \Box^2(r \rightarrow \Box^2 q_i) \land \bigwedge_{1 \leq i \leq 5} \Box^2(q_i \land r) \land \bigwedge_{1 \leq i \neq j \leq 5} (q_i \land q_j)) \]

These formulae will play a similar role to that in the previous section. Indeed, for them we have (cf. Lemma 1).

Lemma 4. For a neighbourhood frame \( F \):

if \( \alpha_{n,k} \) (\( n \geq 0, k \geq 3 \)), \( \beta, \gamma \in E(F) \), then \( \Box p \rightarrow p \in E(F) \).

Let \( b_i, i = 1, 2, 3, 4, 5 \), be arbitrary but fixed elements not belonging to the set of natural numbers \( N \), and let \( (a_n)_{n=1}^\infty \) be a one-to-one sequence of such elements. For any sequence \( m = (m_i)_{i=1}^\infty \) of natural numbers such that \( m_1 = 2, m_i < m_{i+1} \) and \( m_{i+1} - m_i \leq 2 \) (\( i \geq 1 \)), let us put

\[ W_m = N \cup \{b_1, b_2, b_3, b_4, b_5\} \cup \{a_n | n \in m\} \] and

\[ R_m = \{(b_i, b_i) | i \in \{1, 2, 3, 4, 5\} \cup \{b_i, b_{i+1}, (b_{i+1}, b_i) | i \in \{1, 2, 3\} \cup \{(b_1, 1) | i \in \{1, 2, 3, 4\} \cup \{(1, b_4), (b_1, b_5), (b_5, b_4), (b_1, b_4), (1, 1)\} \cup \{(n, m) | n < m \} \cup \{(n, m) | n \geq m \} \cup \{(n, m) | n < m \} \cup \{(n, m) | m < n \} \cup \{(n, m) | m = n \} \cup \{(n, m) | m = n \} \cup \{(a_n, n) \cup \{(a_n, a_n) | n \in m \} \cup \{(a_n, a_n) | n \in m \}. \]

Given \( (W_m, R_m) \),
let $B_m$ denote the modal algebra of finite and cofinite subsets of $W_m$ in which the operation $\Box_m$ corresponding to the connective $\Box$ is defined with the aid of $R_m$, i.e. $\Box_m(S) = \{x \in W_m | \forall y (xR_my \Rightarrow y \in S)\}$.

**Lemma 5.** For each algebra $B_m$

i) $\alpha, \beta, \gamma \in E(B_m)$ ($n \geq 0, k \geq 3$)

ii) if $B \in V(B_m)_{SI}$ and $\Box p \rightarrow p \in E(B)$, then $B \cong 2$.

Lemma 4 and 5 allow us to obtain the following

**Corollary 2.** Let $L$ be a modal logic such that $\Box p \rightarrow p \in L$. Then, for every algebras $B_m$ and neighbourhood frame $E$, $E^+ \in V(K_L \cup \{B_m\})$ iff $E^+ \in K_L$.

Each of the algebras $B_m$ is subdirectly irreducible and $\Box p \rightarrow p \not\in E(B_m)$. Applying the method due to Blok [1] one can prove.

**Lemma 6.** For a modal logic $L$:

if $\Box p \rightarrow p \in L$ then $V(K_L \cup \{B_m\}) \neq V(K_L \cup \{B_n\})$ for every $m \neq n$.

Corollary 2 and Lemma 6 yield the second part of Theorem 1.

§3. We say that a modal logic $L$ is complete with respect to neighbourhood semantics iff $L = \bigcap \{E(E^+) | E^+ \in K_L\}$. We can neither prove nor disprove the following statement (comp. Lemma 4.1 [1]): $L$ is complete with respect to neighbourhood semantics iff $K_L = V(\{E^+ \in K_L | E^+ \text{ is subdirectly irreducible}\})$. If it were proved true, then Theorem 1 would follow immediately from Blok [1].

**References**


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