ON THE DEGREE OF INCOMPLETENESS OF MODAL LOGICS (ABSTRACT)

In the following we will use the well-known correspondence between modal logics and varieties of modal algebras in our investigation of the function which assigns to a modal logic its degree of incompleteness. A modal algebra is an algebra \( A = (A, +, \cdot, ^o, 0, 1) \) where \((a, +, \cdot, 0, 1)\) is a Boolean algebra and \( ^o \) is a unary operation satisfying \( 1^o = 1 \) and \( (x \cdot y)^o = x^o \cdot y^o \); \( ^o \) is called a modal operator. A variety of algebras is a class of algebras closed under the operations of forming homomorphic images, subalgebras as and direct products, and if \( K \) is a class of algebras then \( V(K) \) denotes the smallest variety containing \( K \). The variety of modal algebras is denoted by \( M \), the subvariety of \( M \) defined by the equation \( x^o \cdot x = x^o \) by \( MR \) and the subvariety of \( M \) defined by the equation \( x^{o^n} = x^{o^{n-1}} \), \( n \) a natural number, by \( M^n \). Here \( x^{o^0} = x \), \( x^{o^n} = (x^{o^{n-1}})^o \), \( n \) a natural number. We write \( \Lambda(K) \) for the lattice of subvarieties of a variety \( K \). If \( K, K' \) are varieties such that \( K \subseteq K' \) but for no variety \( K'' \) \( K \nsubseteq K'' \nsubseteq K' \) then we say that \( K' \) is a cover of \( K \). The smallest normal modal logic – containing the axiom \( \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q) \) and closed under the inference rules of modus ponens, substitution and necessitation – will be denoted by \( K \); the lattice of modal logics which are extensions of \( K \) by \( \Lambda(K) \). There is an obvious translation which assigns to any modal formula \( \varphi \) an \( M \)-polynomial \( \hat{\varphi} \). The mapping

\[ \ast : \Lambda(K) \rightarrow A(M) \]

defined by

\[ L \rightarrow L^* = \{ A \in M | A \models \hat{\varphi} = 1, \varphi \in L \} \]

establishes an anti-isomorphism. In particular, \( T^* = MR \) (where \( T \) is the modal logic axiomatized, relative to \( K \), by \( \Box p \rightarrow p \)) and \( SA^* = MR \cap M^2 = \)

---

**Bulletin of the Section of Logic**  
reedition 2011 [original edition, pp. 167–175]  

W. J. Blok
If $F = (W, R)$ is a Kripke frame (i.e., $W$ is a non-empty set and $R$ is a binary relation on $W$) then $F^+$ will be used to denote the modal algebra of all subsets of $W$, the modal operator $^\circ$ being defined by $A^\circ = \{ w \in W | \forall v \in W [(w, v) \in R \Rightarrow v \in A]\}$. A modal algebra isomorphic to one of this kind is called a Kripke algebra. If $K$ is a class of modal algebras then $\mathcal{K}_K$ denotes the subclass of its Kripke algebras. It is a simple matter to verify that for any modal formula $\varphi$ and Kripke frame $F$ we have $F \models \varphi$ iff $F^+ \models \hat{\varphi} = 1$. In accordance with the usual terminology for modal logics we call a variety $K \subseteq \mathcal{M}$ complete iff $K = \mathcal{V}(K_K)$. Clearly, for any modal logic $L \in \Lambda(K)$, $L$ is complete with respect to the Kripke semantics iff $L^*$ is a complete variety.

**Definition.** If $K \in \Lambda(M)$, the degree of incompleteness of $K$, denoted by $\delta(K)$ is

$$|\{K' \in \Lambda(M) | K'_K = K_K\}|.$$ 

Thus for any $K \in \Lambda(M)$, $1 \leq \delta(K) \leq 2^{2^{20}}$. The definition is nothing but an algebraic reformulation of the notion of degree of incompleteness of a modal logic, as introduced in [6] by K. Fine. He presented an example (as S. K. Thomason did in [9]) of a modal logic having degree of incompleteness $\geq 2$. We showed in [2] that every modal logic satisfying certain mild assumptions has degree of incompleteness $2^{20}$. We will now describe the behavior of the function $\delta$ in full detail.

For this we need the notion of splitting algebra, dealt with extensively in [4].

**Definition.** Let $K \subseteq \mathcal{M}$ be a variety. A finite subdirectly irreducible algebra $A \in K$ is called a splitting algebra in $K$ if there exists a variety $K_A \in \Lambda(K)$ such that for every $K' \in \Lambda(K)$ either $A \in K'$ or $K' \subseteq K_A$. If $A$ is a splitting algebra then the variety $K_A$ is called a splitting variety and is denoted by $K/K_A$.

It is easy to see that if a variety $K \subseteq \mathcal{M}$ is generated by its finite members i.e., if the corresponding modal logic has the finite model property then every splitting of $\Lambda(K)$ is determined by some splitting algebra. Each splitting variety $K/K_A$ is definable, relative to $K$, by a single equation $\varepsilon_A = 1$. In [4] we showed that in $MR^2$ every finite subdirectly irreducible algebra or splitting in $MR^2$; this result generalizes to the setting of $\mathcal{M}^\omega$. 

$MR^2$. 

for any natural number $n$, as shown by W. Rautenberg [8]. Some examples in [3] showed that in $MR$ not every finite subdirectly irreducible algebra is splitting.

**Theorem 1.** The only splitting algebra in $MR$ is the two element modal algebra $2 = \{0, 1\}$, with $0^0 = 0$, $1^0 = 1$.

**Theorem 2.** An algebra $A \in M$ is splitting in $M$ iff $A$ is finite, subdirectly irreducible and satisfies $0^{n} = 1$ for some natural number $n$.

The smallest splitting algebra in $M$ is the two element modal algebra $2^+ = \{0, 1\}$, with $0^0 = 1$, $1^0 = 1$. The variety $M/2^+$ is defined by the equation $0^n = 0$ and corresponds with the modal logic $D$ axiomatized relative to $K$ by $\Box T$. The variety $M/2^+$ is the smallest splitting variety, and apparently does not contain any algebras which are splitting in $M$: $2$ is the only splitting algebra in $M/2^+$. Note also that through the class of splitting algebra is rather restricted, it generates $M$.

**Lemma 3.** Let $K \subseteq M$ be a variety satisfying the equation $0^n = 1$ for some natural number $n$. Then the finitely generated algebras in $K$ are finite; i.e., $K$ is locally finite.

Using this lemma we obtain:

**Theorem 4.** Let $\{A_i|i \in I\}$ be a set of splitting algebras in $M$. Then $\bigcap_{i \in I} M/A_i$ is generated by its finite members.

It follows that varieties of this form are complete.

**Theorem 5.** Let $\{A_i|i \in I\}$ be a set of splitting algebras. Then $\delta(\bigcap_{i \in I} M/A_i) = 1$.

**Proof.** Let $K = \bigcap_{i \in I} M/A_i$, $K' \in A(M)$ such that $K_K = K'_K$. Since $K$ is complete, $K' \supseteq K$. Since $A_i$ is finite, $i \in I$, it is a Kripke algebra, hence $A_i \not\subseteq K'$, $i \in I$, thus $K' \subseteq \bigcap_{i \in I} M/A_i = K$.

**Corollary 6.** There are $2^{\aleph_0}$ varieties of modal algebras having degree of incompleteness 1.
Apparently our conjecture in [1] and [2] that every modal logic which is a proper extension of $K$ has degree of incompleteness $2^\aleph_0$ is false. We will now proceed to show, however, that the varieties mentioned in Theorem 5 are the only ones having degree of incompleteness $< 2^\aleph_0$.

A variety is called *tabular* if it is generated by a finite algebra. In [4] it was shown that in $MR^2$ every tabular variety is covered by tabular varieties only, and only by finitely many.

**Theorem 7.** In $MR^3$ the variety $V(2)$ is covered by $2^{\aleph_0}$ varieties.

In terms of modal logics this theorem claims that there are $2^{\aleph_0}$ modal logics containing the axioms $\square p \rightarrow p$ and $\square^2 p \rightarrow \square^3 p$, which are immediate predecessors of classical logic, axiomatized by $\square p \leftrightarrow p$. In particular, these logics need not be tabular. A variety is called *pretabular* if all its proper subvarieties are tabular. A well-known result, proved by several authors (see [7], [5]), states that $MR^2$ contains only five pretabular varieties.

**Corollary 8.** $MR^3$ contains $2^{\aleph_0}$ pretabular varieties.

Note that this result and the previous one are in contradiction with the results claimed in [8], section 3.

Using the varieties produced in the proof of Theorem 7 we are able to prove:

**Theorem 9.** Let $K \in \mathcal{A}(M)$ be a non-trivial variety and $A$ a finite subdirectly irreducible algebra which is not splitting, such that $A \not\in K$ but such that all other homomorphic images of subalgebras of $A$ do belong to $K$. Then $\delta(K) = 2^{\aleph_0}$ and $K$ has $2^{\aleph_0}$ covers in $\mathcal{A}(M)$.

Using Theorem 9 and Theorem 5 we infer:

**Corollary 10.** If $K \in \mathcal{A}(M)$ is such that $K$ is not an intersection of splitting varieties, then $\delta(K) = 2^{\aleph_0}$.

Thus, if $K \in \mathcal{A}(M)$, then $\delta(K) = 1$ if $K$ is an intersection of splitting varieties, otherwise $\delta(K) = 2^{\aleph_0}$. The proofs of theorems provide somewhat sharper results. In order to formulate them the following definition is useful.

**Definition.** For $K \in \mathcal{A}(M)$, $K' \in \mathcal{A}(K)$ let

$$\delta_K(K') = |\{K'' \in \mathcal{A}(K) | K''_K = K'_K\}|.$$
It follows from the constructions that

**Corollary 11.**

(i) \( \delta_{M/2^+}(K) = 2^{\aleph_0} \), for every \( K \in \Lambda(M/2^+) \), such that \( K \) is nontrivial and \( K \neq M/2^+ \).

(ii) \( \delta_{MR}(K) = 2^{\aleph_0} \), for every \( K \in \Lambda(MR) \) such that \( K \) is nontrivial and \( K \neq MR \).

Hence, every proper extension of the modal logic \( T \) has degree of incompleteness (relative to \( T \)) \( 2^{\aleph_0} \). Since in \( M^n \), \( n \) a natural number, every finite subdirectly irreducible algebra is splitting, and hence the \( M^n \) contain many varieties which are intersections of splitting varieties, the function \( \delta_{M^n} \) assumes the value 1 very often. Much more we do not know about the \( \delta_{M^n} \), \( n \geq 3 \). For example, if \( K \in \Lambda(MR^n) \) is non-trivial and tabular, what is \( \delta_{MR^n}(K) \), \( n \geq 3 \)?

A bit more can be said in case \( n = 2 \). Indeed, \( \delta_{MR^2}(K) = 1 \), for every tabular variety \( K \), and more generally, for every variety \( K \subseteq MR^2/F_n^+ \), where \( F_n = \{0,1,\ldots,n-1\}, \leq \). However, as Fine’s example shows [6], there exists a \( K \in \Lambda(MR^2) \) such that \( \delta_{MR^2}(K) \geq 2 \).

As a byproduct we obtain interesting results on the covering relation in \( \Lambda(M) \).

**Definition.** If \( K \in \Lambda(M) \), \( K' \subseteq \Lambda(K) \), let \( c_K(K') = |\{ K'' \in \Lambda(K) | K'' \text{ covers } K' \}| \). We write \( c(K) \) for \( c_M(K) \).

If \( K \in \Lambda(M) \) and \( K' \in \Lambda(K) \), \( K' \neq K \), are such that \( K \) is generated by its finite members or \( K' \) is finitely axiomatizable then \( c_K(K') \geq 1 \). In [3] we gave examples of varieties \( K, K' \) such that \( K' \not\subseteq K \) and \( c_K(K') = 0 \).

**Theorem 13.**

(i) Suppose \( K \in \Lambda(M) \), \( K \neq M \), is an intersection of splitting varieties.

If \( m \) is the smallest cardinal number such that \( K = \bigcap_{i \in I} M/A_i \), \( |I| = m \)

\( m \), for a set of splitting varieties \( \{ M/A_i | i \in I \} \), then \( c(K) = m \).

Hence \( 1 \leq m \leq \aleph_0 \) in this case.

(ii) If not, then \( c(K) = 2^{\aleph_0} \) if \( K \) is non-trivial; \( c(K) = 2 \) if \( K \) is trivial.
Theorem 14.

(i) For every $K \in \Lambda(M/2^+)$, $K$ nontrivial and $K \neq M/2^+$, $c_{M/2^+}(K) = 2^{ℵ_0}$.

(ii) For every $K \in \Lambda(MR)$, $K$ nontrivial, $K \neq MR$, $c_{MR}(K) = 2^{ℵ_0}$.

Corollary 15. For $K \in \Lambda(M)$, $K$ non-trivial, $K \neq M$, $δ(K) = 1$ iff $c(K) \leq ℵ_0$ and $δ(K) = 2^{ℵ_0}$ iff $c(K) = 2^{ℵ_0}$.

References


Mathematisch Instituut
Roetersstraat 15
Amsterdam
The Netherlands