Tadeusz Prucnal

ON FRIEDMAN’S PROBLEM IN MATHEMATICAL LOGIC*
(Preliminary Report)

0. Let $F = \langle F, \wedge, \sim, \square \rangle$ be the free algebra in the class of all algebras of the type $\langle 2, 1, 1 \rangle$ free-generated by the set $V = \{p_1, p_2, \ldots \} = \{p_i : i \in N \}$. By $h^e$ we denote the extension of the function $e : V \rightarrow F$ to the endomorphism of the algebra $F$.

H. Friedman in [1] conjectured that there are sets $M \subseteq F$ such that:

(F1) $V \subseteq M$
(F2) $\sim \alpha \in M \iff \alpha \not\in M$
(F3) $\alpha \land \beta \in M \iff \alpha \in M \land \beta \in M$
(F4) $\square \alpha \in M \iff \bigvee_{e : V \rightarrow F} h^e(\alpha) \in M$

for every $\alpha, \beta \in F$.

In this paper we will show that there exists a set $M \subseteq F$ such that the conditions (F1) – (F4) are satisfied.

We shall use the symbols: $\iff, \Rightarrow, \land, \lor$ as the well-known propositional connectives from metalanguage. The symbols $\forall$ and $\exists$ will also be used as quantifiers from metalanguage.

1. Let now $F = \langle F, \lor, \land, \rightarrow \rangle$ be the free algebra in the class of all algebras of the type $\langle 2, 2, 2, 1 \rangle$ free-generated by the set $V$. By $T$ we denote the well-known McKinsey-Tarski transformation (cf. [2]), which maps $F$ into $F$ in the following way:

a. $T(p_i) = \square p_i$
b. $T(\sim \alpha) = \square \sim T(\alpha)$

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c. $T(\alpha \land \beta) = T(\alpha) \land T(\beta)$,
d. $T(\alpha \lor \beta) = \sim [\sim T(\alpha) \land \sim T(\beta)]$.
e. $T(\alpha \rightarrow \beta) = \Box [\sim T(\alpha) \land \sim T(\beta)]$

for every $i \in \mathbb{N}$ and $\alpha, \beta \in F$.

By $\text{INT}$ we mean the set of all theorems of intuitionistic propositional logic, and by $\text{S}4$ – the set of all theorems of modal logic. $\text{Cn}_{\text{INT}}(X)$ is the smallest set containing $\text{INT} \cup X \subseteq F$ and closed under the modus ponens rule. Similarly: $\text{Cn}_{\text{S}4}(Y)$ is the least set containing $Y \cup \text{S}4 \subseteq F$ and closed under the modus ponens rule: $\sim (\alpha \land \sim \beta), \alpha/\beta$.

We have:

**Lemma 1.** (Cf. [7]). For every $\gamma \in F$ and $X \subseteq F$:

$$\gamma \in \text{Cn}_{\text{INT}}(X) \Leftrightarrow T(\gamma) \in \text{Cn}_{\text{S}4}(T(X)),$$

where $T(X)$ is the image of the set $X$.

By $\text{FT}$ we denote the least set containing the image $T(F)$ and closed with respect to: $\land, \sim$, and $\square$.

**Lemma 2.** (Cf. [6]). For every $\alpha \in \text{FT}$ there are $\gamma_1, \gamma_2, \ldots, \gamma_n, \delta_1, \delta_2, \ldots, \delta_n \in F$ such that:

$$\alpha \equiv \sim [T(\gamma_1) \land \sim T(\delta_1)] \land \ldots \land [T(\gamma_n) \land \sim T(\delta_n)],$$

where $\alpha \equiv \beta \Leftrightarrow \sim (\alpha \land \sim \beta) \land \sim (\beta \land \sim \alpha) \in \text{S}4$.

Let $B$ be the least set containing $\{\sim p_i : i \in \mathbb{N}\}$ and closed under the connectives: $\lor, \land, \rightarrow, \sim$.

Putting

$$\text{ML} =_{df} \{\gamma \in F : \forall e. \sim B h^e(\gamma) \in K P\},$$

where $K P$ is an intermediate logic obtained by adding to $\text{INT}$ the axioms: $(\sim \gamma \rightarrow \alpha \lor \beta) \rightarrow (\sim \gamma \rightarrow \alpha) \lor (\sim \gamma \rightarrow \beta), \alpha, \beta \in F$, we obtain an intermediate logic such that $K P \nsubseteq \text{ML}$.

We have:

**Lemma 3.** (Cf. [3]). For every $\alpha, \beta \in F$:

$$\alpha \lor \beta \in \text{ML} \Leftrightarrow \alpha \in \text{ML} \lor \beta \in \text{ML}.$$
This $ML$ has also the following property¹:

**Lemma 4.** (Cf. [5]). For every $\alpha, \beta \in F$:

$$\alpha \rightarrow \beta \in ML \iff \forall e: V \rightarrow F[h^e(\alpha) \in ML \Rightarrow h^e(\beta) \in ML].$$

Putting

$$ML(T) =_{df} CnS_4(T(ML)),$$

we obtain:

**Lemma 5.**

(i) $\gamma \in ML \iff T(\gamma) \in ML(T)$, for every $\gamma \in F$.

(ii) $\alpha \in ML(T) \iff \Box \alpha \in ML(T)$, for every $\alpha \in F\Box$.

(iii) $\sim (\sim \Box \alpha \land \sim \Box \beta) \in ML(T) \iff \Box \alpha \in ML(T) \lor \Box \beta \in ML(T)$, for every $\alpha, \beta \in F_T$.

Let now $ML\Box$ be a set defined as follows:

$$ML\Box =_{df} \{\alpha \in F\Box : \forall e: V \rightarrow F_T h^e(\alpha) \in ML(T)\}.$$

**Lemma 6.** For every $\gamma \in F$:

$$\gamma \in ML \iff T(\gamma) \in ML\Box.$$

**Lemma 7.** For every $\alpha, \beta \in F\Box$:

(i) $S4 \subsetneq ML\Box$,

(ii) $\alpha, \sim (\alpha \land \sim \beta) \in ML\Box \Rightarrow \beta \in ML\Box$,

(iii) $\alpha \in ML\Box \Rightarrow \forall e: V \rightarrow F_T h^e(\alpha) \in ML\Box$,

(iv) $\alpha \in ML\Box \iff \Box \alpha \in ML\Box$,

(v) $\Box \alpha \in ML\Box \lor \Box \beta \in ML\Box \iff \sim (\sim \Box \alpha \land \sim \Box \beta) \in ML\Box$.

We define now a set $A \subseteq F\Box$ in the following way:

$$\beta \in A \iff \exists e: V \rightarrow F_T \exists \alpha \in F - ML\Box \beta = \sim (\Box \alpha \land h^e(\alpha)),$$

for every $\beta \in F\Box$.

¹Let us note that Lemma 4 states that the calculus $ML$ is structurally complete in the sense of W. A. Pogorzelski [4].
Thus:

**Lemma 8.** \( Cn_{S4}(ML\Box \cup A \cup V \cup \{\sim T(\gamma) : \gamma \in F - ML\}) \neq F\Box. \)

Let \( M_0 =_{df} Cn_{S4}(ML\Box \cup A \cup V \cup \{\sim T(\gamma) : \gamma \in F - ML\}) \) and let \( M_* \) be the maximal element in \( \{M \subseteq F\Box : M_0 \subseteq M = Cn_{S4}(M) \neq F\Box\} \). Thus we have:

**Theorem.** The set \( M_* \) satisfies the conditions \((F1) - (F4)\).

References


*Institute of Mathematics
Pedagogical College, Kielce*

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Dr J. Perzanowski informed me that an analogous results had been obtained by Kit Fine (unpublished).