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ON SOME INTUITIONISTIC MODAL LOGICS

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Some modal logics based on logics weaker than the classical logic have been studied by Fitch [4], Prior [7], Bull [1], [2], [3], Prawitz [6] etc. Here we treat modal logics based on the intuitionistic propositional logic, which call intuitionistic modal logics (abbreviated as IML's).

Let $H$ be the intuitionistic propositional logic formulated in the Hilbert-style. The rules of inference of $H$ are modus ponens and the rule of substitution. The IML $L_0$ is obtained from $H$ by adding the following three axioms,

\[ \Box p \supset p, \]
\[ p \supset \Box \Box p, \]
\[ (p \supset q) \supset (\Box p \supset \Box q), \]

and the rule of necessitation, i.e., from $A$ infer $\Box A$. It is clear that $L_0$ with the law of excluded middle becomes $S4$. Now, consider the following axioms.

\[ A_1 : \neg \Box p \supset \Box \neg \Box p, \]
\[ A_2 : (\Box p \supset \Box q) \supset \Box (\Box p \supset \Box q), \]
\[ A_3 : \Box (\Box p \lor q) \supset (\Box p \lor \Box q), \]
\[ A_4 : \Box p \lor \Box \neg \Box p. \]

The logic $L_0$ with the axiom $A_i$ is denoted by $L_i$ for $i = 1, 2, 3, 4$. The logic $L_3$ with $A_1$ (or $A_2$) is denoted by $L_{31}$ (or $L_{32}$). It is easy to see that $S4$ with any $A_i$ is equal to $S5$.

We identity a logic $L$ with the set of formulas provable in $L$. 

Theorem 1.

(i) For $J = 1, 2, 3, 31, 32$, $L_0 \subsetneq L_J \subsetneq L_4$.
(ii) $L_1 \subsetneq L_2 \subsetneq L_{32}$ and $L_3 \subsetneq L_{31} \subsetneq L_{32}$.

For IML’s, we introduce a kind of Kripke models, which we call I models. A triple $(M, \leq, R)$ is an I frame, if 

(i) $M$ is a nonempty set with a partial order $\leq$,
(ii) $R$ is a reflexive and transitive relation on $M$ such that $x \leq y$ implies $xRy$ each $x, y \in M$.

For any formula $A$ and an element $a \in M$, a valuation $W(A, a) \in \{t, f\}$ is defined in the same way as a valuation on a Kripke model for the intuitionistic propositional logic. For instance,

$W(A \supset B, a) = T$ if and only if for any $b$ such that $a \leq b, W(A, b) = f$ or $W(B, b) = t$.

Moreover, we claim that $W(\Box A, a) = t$ if and only if for any $b$ such that $a \sim R b W(A, b) = t$.

A quadruple $(M, \leq, R, W)$ is an I model if $(M, \leq, R)$ is an I frame and $W$ is a valuation on it. A formula $A$ is valid in an I frame $(M, \leq, R)$ if $W(A, a) = t$ for any valuation $W$ on $(M, \leq, R)$ and any element $a \in M$.

For any binary relation $R$, we write $x \sim_R y$ if $xRy$ and $yRx$ hold. In what follows we omit the subscript letter $R$. Now define I frames of type $J$ for $J = 0, 1, 2, 3, 31, 32, 4$ as follows.

(0) Any I frame is of type 0.
(1) An I frame $(M, \leq, R)$ is of type 1 when for each $x, y \in M$, if $xRy$ then there is an element $y'$ in $M$ such that $x \leq y'$ and $yRy'$.
(2) An I frame $(M, \leq, R)$ is type 2 when for each $x, y \in M$, if $xRy$ then there is an element $y'$ in $M$ such that $x \leq y'$ and $y \sim y'$.
(3) An I frame $(M, \leq, R)$ is of type 3 when for each $x, y \in M$, if $xRy$ then there is an element $x'$ in $M$ such that $x \sim x'$ and $x' \leq y$.
(3i) An I frame is of type 3i if it is both of type 3 and of type $i$, for $i = 1, 2$.
(4) An I frame $(M, \leq, R)$ is of type 4 if $R$ is symmetric.
Theorem 2. A formula is provable in $L_J$ if and only if it is valid in any $I$ frame of type $J$, for $J = 0, 1, 2, 3, 31, 32, 4$.

An IML $L_J$ has the finite model property if for any formula $A$ not provable in $L_J$ there is a finite $I$ frame of type $J$ in which $A$ is not valid.

Theorem 3. For $J = 0, 2, 3, 32, 4$, $L_J$ has the finite model property.

In [5], another kind of Kripke models is introduced and discussed.

References