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A NEW PROOF OF STRUCTURAL COMPLETENESS OF LUKASIEWICZ’S LOGICS

The problem of structural completeness of the finite-valued Lukasiewicz’s sentential calculi was investigated and solved in [4], [7], [6]. The present paper contains a new proof of all these results.

I. Let \( \langle S; F_1, \ldots, F_n \rangle \) be a propositional language. A matrix \( M = \langle |M|, |M|^*; f_1, \ldots, f_n \rangle \) of this language is embeddable in a matrix \( N = \langle |N|, |N|^*; g_1, \ldots, g_n \rangle \) (\( M \subseteq N \)) iff there exists a monomorphism \( h : M \to N \). The symbol \( N \times M \) stands for the product of matrices and the structural consequence generated by \( M \) (cf. [1]) is denoted by \( \overrightarrow{M} \). We know (cf. [9]) for every \( M, N \):

\[
(1.1) \quad N \subseteq M \Rightarrow \overrightarrow{M} \leq \overrightarrow{N}.
\]

\[
(1.2) \quad \overrightarrow{M} = \overrightarrow{M/\sim} \quad \text{for every congruence } \sim \text{ of matrix } M.
\]

Moreover, one can prove (cf. [11]) that

\[
(1.3) \quad \overrightarrow{M \times N}(X) = \begin{cases} \overrightarrow{M}(X) \cap \overrightarrow{N}(X) & \text{if } X \in \text{Sat}(M) \quad \text{and} \quad X \in \text{Sat}(N) \\ S & \text{if } X \notin \text{Sat}(M) \quad \text{or} \quad X \notin \text{Sat}(N) \end{cases}
\]

We recall that \( X \in \text{Sat}(M) \) iff there exists \( v : At \to |M| \) such that \( h^v(X) \subseteq |M|^* \). A consequence \( Cn \) (or a propositional calculus \( \langle R, A \rangle \)) is structurally complete iff every structural and permissible rule of this consequence (of this calculus) is at the same time a derivable rule of \( Cn \) (of \( \langle R, A \rangle \)). Let \( L_{Cn(0)} \) denotes the Lindenbaum’s matrix \( \langle S, Cn(O); F_1, \ldots, F_n \rangle \) of a consequence \( Cn \).

The following theorem characterize a structurally complete consequence in the set of all structural consequences:
\begin{align}
(1.4) \quad & Cn \in \text{Struct} \Rightarrow [Cn \in \text{SCpl} \Leftrightarrow Cn = \overrightarrow{L}_{Cn(0)}] \\
(1.5) \quad & Cn \in \text{Struct} \Rightarrow \{Cn \in \text{SCpl} \Leftrightarrow \forall_{Cn_1 \in \text{Struct}}[Cn_1(0) = Cn(0) \Rightarrow Cn_1 \leq Cn]\} \\
\end{align}

Theorem (1.4) is proved in [5] and (1.5) can be found in [2]. Let $Sb$ be the consequence operation based on the substitution rule only. It is easy to see that:

\begin{align}
(1.6) \quad & Cn \in \text{Struct} \cap \text{SCpl} \Rightarrow Cn_{Sb} \in \text{SCpl}.
\end{align}

From the finitionistic point of view by a rule of inference we mean an operation with finite set of premises. Then we must consider finitionistic structural completeness. For finitionistic structural completeness (to be denoted by $\text{SCpl}_F$) it can be proved (cf. [2]) that:

\begin{align}
(1.7) \quad & Cn \in \text{Struct} \Rightarrow \{Cn \in \text{SCpl}_F \Leftrightarrow \forall_{Cn_1 \in \text{Struct} \cap \text{Fin}}[Cn_1(O) \Rightarrow Cn_1 \leq Cn]\}
\end{align}

II. In the sequel, the symbol $S$ will denote the set of formulas built up by means of propositional variables and the following connectives (notation follows that of Lukasiewicz): $C$ (implication), $K$ (conjunction), $E$ (equivalence), $A$ (disjunction) and $N$ (negation). The symbol $S^p$ stands for the set of all positive formulas. All results from I will be applied to these language. Let $M_n = \langle\{0, \frac{1}{n-1}, \ldots, 1\}, \{1\}; c, a, k, e, n\rangle$ be the $n$-valued Lukasiewicz’s matrix and let $M^p_n = \langle|M_n|, \{1\}; c, a, k, e\rangle$ be a positive reduct of this matrix. Moreover, we put $R_0^* = \{r_0, r^*_s\}$ and $R_0 = \{r_0\}$, where $r_0$ is the modus ponens rule and $r^*_s$ is the substitution rule. It is known (cf. [8]) that for $(R_0^*, A_n)$ (the $n$-valued Lukasiewicz’s calculus) and for $(R_0^*, A^p_n)$ (the positive $n$-valued Lukasiewicz’s calculus) we have:

\begin{align}
(2.1) \quad & Cn(R_0, Sb(A_n) \cup X) = \overrightarrow{M}_n(X) \text{ for every } X \subseteq S. \\
& Cn(R_0, Sb(A^p_n) \cup X) = \overrightarrow{M}^p_n(X) \text{ for every } X \subseteq S^p.
\end{align}

From now onward, the symbol $L_n$ will denote the Lindenbaum’s matrix of $(R_0^*, A_n)$ and $L^p_n$ will be the Lindenbaum’s matrix of $(R_0^*, A^p_n)$. Let us define a relation $\approx_n$ on $S$:

\begin{align}
(2.2) \quad & \alpha \approx_n \beta \Leftrightarrow E\alpha \beta \in E(M_n).
\end{align}

From this definition it directly follows that $\alpha \approx_n \beta$ iff $h^n\alpha = h^n\beta$ for every
v : At → |Mn|. The relation ≈ is a congruence of the Lindenbaum’s matrix Lₙ. By induction one can prove that for every v : At → |Mn| and for every formula Cₖpq (Cₖ₊₁pq = CpCₖpq)

(2.3) hᵥ(Cₖpq) = min{1, k(1 – vp) + vq}

Hence we obtain that for every v : At → |Mn|

(2.4) hᵥ(Cₙ⁻¹pq) = \begin{cases} 1 & \text{if } vp \neq 1 \\ vq & \text{if } vp = 1 \end{cases}

(2.5) hᵥ(Cₙ⁻²pq) = \begin{cases} vq & \text{if } vp = 1 \\ vp & \text{if } vp = \frac{n-2}{n-1} \text{ and } vq = 0 \\ 1 & \text{otherwise.} \end{cases}

Since in every Mₙ we have hᵥ(CCPPq) = max{vp, vq} = a(vp, vq) we infer from (2.4) that:

(2.5) hᵥ(CCCCₙ⁻²pqpp) = \begin{cases} \frac{n-2}{n-1} & \text{if } vp = \frac{n-2}{n-1} \text{ and } vq = 0 \\ 1 & \text{if } vp \neq \frac{n-2}{n-1} \text{ or } vq \neq 0 \end{cases}

(2.5) hᵥ(CCCCₙ⁻¹Cₙ⁻²pqCCₙ⁻¹pq) = \begin{cases} 0 & \text{if } vp = \frac{n-2}{n-1} \text{ and } vq = 0 \\ 1 & \text{if } vp \neq \frac{n-2}{n-1} \text{ or } vq \neq 0 \end{cases}

III. We shall prove now the following fundamental theorem on structural completeness of Lukasiewicz’s logics.

(3.1) Mₙ × M₂ ∈ SCpl for every n > 2.

PROOF. Suppose that n > 2 is a fixed natural number. For every x ∈ |Mₙ| let αₓ₁, αₓ² be two formulas defined as follows:

αₓ₁ = CPP, αₓ² = CCCₙ⁻²pqpp,

αₓ₀ = CCCCₙ⁻¹Cₙ⁻²pqCCₙ⁻¹pq (if n > 2),

αₓ = Cₓ⁻¹αₓ₋₁, αₓ for 0 ≤ x < \frac{n-2}{n-1}, αₓ₀ = Nαₓ₋₁₀.

From (2.3) and (2.5) it follows that for every v : At → |Mn|

hᵥ(αₓ) = \begin{cases} i & \text{if } vp \neq \frac{n-2}{n} \text{ or } vq \neq 0 \\ x & \text{if } vp = \frac{n-2}{n} \text{ and } vq = 0 \end{cases}
It is easy to verify that for every $v : At \to |M_n|$

$$h^v(Ca_x^i \alpha_{y}^j) = h^v(\alpha_{e(x,y)}^{e(i,j)}), \quad h^v(Aa_x^i \alpha_{y}^j) = h^v(\alpha_{a(x,y)}^{a(i,j)}),$$

$$h^v(Ka_x^i \alpha_{y}^j) = h^v(\alpha_{k(x,y)}^{k(i,j)}), \quad h^v(Ea_x^i \alpha_{y}^j) = h^v(\alpha_{e(x,y)}^{e(i,j)}),$$

$$h^v(Na_x^i) = h^v(\alpha_{n(i)}^{n(i)}).$$

Hence, $Ca_x^i \alpha_{y}^j \equiv_n \alpha_{e(x,y)}^{e(i,j)}, \quad Aa_x^i \alpha_{y}^j \equiv_n \alpha_{a(x,y)}^{a(i,j)}$, $Ka_x^i \alpha_{y}^j \equiv_n \alpha_{k(x,y)}^{k(i,j)}, \quad Ea_x^i \alpha_{y}^j \equiv_n \alpha_{e(x,y)}^{e(i,j)}, \quad Na_x^i \equiv_n \alpha_{n(i)}^{n(i)}$.

From the above it follows that a mapping $f : |M_n| \times |M_2| \to |L_n/\approx_n|$ defined as follows $f(\langle x, i \rangle) = [\alpha_x^i]$ is a monomorphism. Hence, $M_n \times M_2 \subseteq L_n/\approx_n$ and from (2.1) and (2.2) we obtain $M_n \times M_2 \triangleright L_n/\approx_n = L_n^\beta$. On the other hand, since $M_n^\beta(0) = L_n^\beta(0) = M_2 \times M_n(0)$ we infer from (1.4) and (1.5) that $M_n \times M_2 \leq L_n^\beta$. Hence $M_n \times M_2 = L_n^\beta \in SCpl$.

Note that this theorem is proved without McNaughton’s criterion and the representation theorem for Lukasiewicz’s algebras. Theorem (3.1) can be immediately deduced from results of [3] and [10]. From (1.3) it follows that

$$M_n \times M_2(X) = \begin{cases} M_n^\beta(X) & \text{if } X \in Sat(M_2) \\ S & \text{if } X \notin Sat(M_2). \end{cases}$$

It is easy to see that $M_n \times M_2 \neq M_n^\beta$ (for example, if $\alpha = C^{n-1}CCNpppNCCNppp$, then $M_n^\beta(\alpha) \neq S$ and $\alpha \notin Sat(M_2)$) for $n > 2$.

From this and (1.7) it follows that (cf. [7]):

(3.2) $\langle R_0, Sb(A_n) \rangle \notin SCpl_F$ for every $n > 2$.

Now we shall prove the Tokarz’s theorem on structural completeness of the Lukasiewicz’s sentential calculi.

(3.3) $\langle R_0, A_n \rangle \in SCpl_F$ for every $n \geq 2$.

**Proof.** To prove that $M_n^\beta(\text{Sb}(X)) = M_n \times M_2(\text{Sb}(X))$ for every $X \subseteq S$ it suffices to show that $M_n^\beta(\text{Sb}(X)) = S$ for every $\text{Sb}(X) \notin Sat(M_2)$. Suppose that it is not true; i.e. there exists $\text{Sb}(X) \notin Sat(M_2)$ such that $M_n(\text{Sb}(X)) \neq S$. Hence there exists $v : At \to |M_n|$ such that $h^v(\text{Sb}(X)) \subseteq$
Let $e : At \rightarrow S$ be a substitution defined as follows: $e(\gamma) = C\gamma\gamma$. We have $w = h^e : At \rightarrow \{0, 1\}$ so that $h^w(Sb(X)) = h^e(h^e(Sb(X))) \subseteq h^e(Sb(X)) \subseteq \{1\}$. Contradiction. Hence: $M_n \times M_2(Sb(X)) = M_n(Sb) = Cu(R_0, Sb(A_n) \cup Sb(X)) = Cu(R_0, A_n \cup X)$. From (1.6) and (3.1) follows that $\langle R_0, A_n \rangle \in SCpl$.

We shall complete the paper by the following theorem:

\begin{equation}
\langle R_0, Sb(A_p^n) \rangle \in SCpl \text{ for every } n \geq 2.
\end{equation}

**Proof.** The proof is similar to that (3.1). Using the notation from (3.1) we have $a_1^x \in S^p$ for every $x \in |M_n|$ and a mapping $g(x) = [a_1^x], g : |M_n| \rightarrow S^p/\approx_n$ is a monomorphism of $M_n^p$ and $L_n^p$. Hence, $M_n^p \supseteq L_n^p/\approx_n = L_n^p$. From (1.4) and (1.5) we obtain $M_n^p \leq L_n^p$. Thus $M_n^p = L_n^p \in SCpl$.

This theorem is a generalization of some theorem from [6], where it is proved that pure implicational Lukasiewicz's calculi are $SCpl_F$.

**References**


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