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SOME REMARKS ON THREE-VALUED IMPLICATIVE SENTENTIAL CALCULI

Let \( \rightarrow, \vee, \wedge, \neg, f \) be functions defined on the set \( \{2, 1, 0\} \) for which the following conditions are satisfied:

\begin{enumerate} 
  \item[(i)] (A) \( (x \rightarrow x) = 2 \)
  \item[(B)] if \( (x \rightarrow y) = 2 \) and \( x = 2 \), then \( y = 2 \)
  \item[(C)] \( (x \rightarrow 2) = 2 \)
  \item[(D)] if \( (x \rightarrow y) = 2 \) and \( (y \rightarrow z) = 2 \), then \( (x \rightarrow z) = 2 \)
  \item[(E)] if \( (x \rightarrow y) = 2 \) and \( (y \rightarrow x) = 2 \), then \( x = y \)
\end{enumerate}

\begin{enumerate} [resume] 
  \item[(ii)] \( x \land y = \max(x, y) \)
  \item[(iii)] \( x \lor y = \min(x, y) \)
  \item[(iv)] \( fx = 0 \)
  \item[(v)] \( \neg x = (x \rightarrow fx) \)
\end{enumerate}

The matrix

\[ Z = \langle \{2, 1, 0\}, \{2\}, \rightarrow, \vee, \land, \neg, f \rangle \]

will be called an \textit{implicative three-valued matrix}.

**Remark 1.** The conditions (A) - (B) are analogous to those given by Rasiowa for the operation \( \rightarrow \) in an implicative algebra (see [1]).

**Remark 2.** There exist 32 mutually different three-valued matrices

\[ Z_{(k)} = \langle \{2, 1, 0\}, \{2\}, \rightarrow, \vee, \land, \neg_k, f \rangle \]

the functions of which satisfy the conditions (i) - (v).

By a \textit{(k)-implicative three-valued sentential calculus} we shall understand any couple \( S_k = (L_k, Cn(k)) \) where \( L_k = \langle L_k, \rightarrow_k, \vee, \land, \neg_k, f \rangle \) is a sentential
language and $Cn_{(k)}$ is a consequence determined by the $(k)$-implicative three-valued matrix $Z_{(k)}$.

From the remark given above it follows that there exists 32 such three-valued calculi. The individual calculi $S_k$ differ from one another in the connectives of implication $\to_k$ and negation $\neg_k$, while the connectives of disjunction, conjunction and falsum are determined by the same tables. The aim of this paper is to compare these calculi with respect to mutual definability of the connectives $\to_k$ and $\neg_k$.

Let $S_k = (L_k, Cn_{(k)})$ be a $(k)$-implicative three-valued sentential calculus. We shall say that $S_k$ is definitionally complete if and only if the $(k)$-implicative matrix $Z_{(k)}$ is functionally complete.

Let $S_k = (L_k, Cn_{(k)})$, $S_i = (L_i, Cn_{(i)})$ be two sentential calculi. $S_k$ will be said to be definable in $S_i$, in symbols

$$S_k \subset \cdot S_i$$

if and only if the connective of implication $\to_k$ (and therefore that of $\neg_k$) is definable in the calculus $S_i$. If $S_k \subset S_i$ and $S_i \subset S_k$ we shall say that $S_k$ and $S_i$ are mutually reconstructable in symbols

$$S_k \sim S_i$$

The symbols $S_p$, $S_L$, $S_H$, $S_1$ will be used to denote definitionally complete calculus, three-valued logic of Lukasiewicz, Heyting’s three-valued calculus and the calculus with an implication $\to_1$ and negation $\neg_1$ being determined by the table

<table>
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<th>1</th>
<th>0</th>
<th>$\neg_1$</th>
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<td>2</td>
<td>1</td>
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</tbody>
</table>

respectively.

Now we define the sets $\overline{S}_p, \overline{S}_L, \overline{S}_H, \overline{S}_1$ as follows:

$$\overline{S}_p = \{S_k : S_k \sim S_p\}$$
$$\overline{S}_L = \{S_k : S_k \sim S_L\}$$
$$\overline{S}_H = \{S_k : S_k \sim S_H\}$$
$$\overline{S}_1 = \{S_k : S_k \sim S_1\}$$

The following theorem holds:
Theorem 2. The mutual reconstructability relation $\sim$ determines a decomposition of the set of 32 calculi into four equivalence classes of $\sim$, namely $\mathcal{S}_p$, $\mathcal{S}_L$, $\mathcal{S}_H$, $\mathcal{S}_1$, such that two calculi $\mathcal{S}_i, \mathcal{S}_j$ belong to the same equivalence class if and only if $\mathcal{S}_i \sim \mathcal{S}_j$.

Let $\mathcal{S}_k, \mathcal{S}_l$ be two equivalence classes. We shall say that $\mathcal{S}_k$ is definable in $\mathcal{S}_l$, in symbols $\mathcal{S}_k \subset \mathcal{S}_l$, if and only if there exist two calculi $\mathcal{S}_j \in \mathcal{S}_k$ and $\mathcal{S}_1 \in \mathcal{S}_l$ such that $\mathcal{S}_j \subset \mathcal{S}_1$.

The following theorem gives relations between $\mathcal{S}_p$, $\mathcal{S}_L$, $\mathcal{S}_H$, $\mathcal{S}_1$.

Theorem 3.

(i) $\mathcal{S}_L \subset \mathcal{S}_p$, $\mathcal{S}_H \subset \mathcal{S}_p$, $\mathcal{S}_1 \subset \mathcal{S}_p$,
(ii) $\mathcal{S}_p \not\subset \mathcal{S}_L$, $\mathcal{S}_p \not\subset \mathcal{S}_H$, $\mathcal{S}_p \not\subset \mathcal{S}_1$,
(iii) $\mathcal{S}_L \not\subset \mathcal{S}_1$, $\mathcal{S}_1 \not\subset \mathcal{S}_L$,
(iv) $\mathcal{S}_H \subset \mathcal{S}_L$, $\mathcal{S}_L \not\subset \mathcal{S}_H$,
(v) $\mathcal{S}_H \not\subset \mathcal{S}_1$, $\mathcal{S}_1 \not\subset \mathcal{S}_H$.

References


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