NOTES ON THE RASIOWA – SIKORSKI LEMMA

This paper aims at formulating a condition necessary and sufficient for the existing of a prime filter preserving enumerable infinite joins and meet in a distributive lattice.

First we prove a simple but very useful Lemma 1. Every lattice \( A = (A, \cup, \cap) \) is distributive if and only if \( a \leq b \) provided that there exists \( c \in A \) such that \( c \neq a, c \neq b \) and \( a \leq b \cup c \) and \( a \cap c \leq b \).

Suppose that \( A \) is a distributive lattice and \( a \leq b \cup c, a \cap c \leq b \) for some \( c \in A, c \neq a, c \neq b \). Then

\[
    a = a \cap a \leq a \cap (b \cup c) = (a \cap b) \cup (a \cap c) \leq (a \cap b) \cup b = b
\]

Other direction, suppose that a lattice \( A \) is not distributive and let \( A \) be the form

\[
    a \bullet \quad c \bullet \quad b
\]

It is obvious that \( a \leq b \cup c \) and \( a \cap c \leq b \) but \( a \leq b \) does not hold.

Corollary. For every lattice \( A = (A, \cup, \cap) \) the next two conditions are equivalent:
(i) \( a \leq b \) if and only if there exists \( c \in A \) such that \( c \neq a, c \neq b, a \leq b \cup c \) and \( a \cap c \leq b \),

(ii) \( A \) is a distributive lattice.

It is well known that

**Lemma 2.** In every lattice \( A \), if the infinite join and meets concerned exist; then

\[
\bigcup_{t \in T} (a_t \cap b) \leq b \cap \bigcup_{t \in T} a_t,
\]

\[
\bigcap_{t \in T} b_t \cup a \leq \bigcap_{t \in T} (b_t \cup a).
\]

Let \( A = (A, \cup, \cap) \) be a lattice and let for every \( n \in \omega \), \( A_{2n} \subset A \) and \( B_{2n+1} \subset A \). We denote

\[
a_{2n} = \bigcup_{a \in A_{2n}} a \quad \quad \quad b_{2n+1} = \bigcap_{b \in B_{2n+1}} b
\]

A prime filter \( \nabla \) is said to be a \( Q \)-filter provided that

(f1) for every \( n \in \omega \) if \( a_{2n} \in \nabla \) then \( A_{2n} \cap \nabla \neq \emptyset \),

(f2) for every \( n \in \omega \) if \( B_{2n+1} \subset \nabla \) then \( b_{2n+1} \in \nabla \).

**Theorem.** Let \( A = (A, \cup, \cap) \) be a distributive lattice and let for every \( n \in \omega \), \( a_{2n} \) and \( b_{2n+1} \) exist. Suppose, that for some \( x, y \) the inequality \( x \leq y \) does not hold.

Then there exists a \( Q \)-filter \( \nabla \) such that \( x \in \nabla \) and \( y \notin \nabla \) if and only if for every \( n \in \omega \), \( a', b' \in A \)

\[
(\cap, \cup) \bigcap_{b \in B_{2n+1}} (a' \cup b) \leq a' \cup \bigcap_{b \in B_{2n+1}} b,
\]

\[
(\cup, \cap) b' \cap \bigcup_{a \in A_{2n}} a \leq \bigcup_{a \in A_{2n}} (b' \cap a).
\]

**Proof.** Suppose that there exist \( n_0 \in \omega \) such that

\[
\bigcap_{b \in B_{2n_0+1}} (a' \cup b) \leq a' \cup \bigcap_{b \in B_{2n_0+1}} b
\]

does not hold. Then by our assumption there exists a \( Q \)-filter \( \nabla \) such that \( \bigcap_{b \in B_{2n_0+1}} (a' \cup b) \in \nabla \) and \( (a' \cup \bigcap_{b \in B_{2n_0+1}} b) \notin \nabla \), for some fixed \( n_0 \in \omega \). Hence \( \nabla \) is a \( Q \)-filter we infer that for every \( b \in B_{2n_0+1}, b \in \nabla \) or \( a' \in \nabla \) and \( a' \notin \nabla \) and there exists \( b'_0 \in B_{2n_0+1} \) such that \( b'_0 \notin \nabla \). But this is a contradiction.
In a similar way we prove the condition \( (\bigcup, \cap) \). If for some \( n_0 \in \omega \)

\[
b' \cap \bigcup_{a \in A_{2n_0}} a \subseteq \bigcup_{a \in A_{2n_0}} (b' \cap a)
\]
does not hold, then there exists a \( Q \)-filter \( \nabla \) such that

\[
b' \cap \bigcup_{a \in A_{2n_0}} a \in \nabla \text{ and } \bigcup_{a \in A_{2n_0}} (b' \cap a) \notin \nabla.
\]

Thus

\[
b' \in \nabla \land \exists a \in A_{2n_0} \ a \in \nabla \land \forall a \in A_{2n_0} \sim (b' \in \nabla \land a \in \nabla)
\]

but it is a contradiction which proves necessity.

Other direction. Suppose that for some \( x, y, x \leq y \) does not hold and the conditions \( (\bigcap, \bigcup) \) and \( (\bigcup, \cap) \) are satisfied.

Now we will construct two sequences \((\alpha_n)_{n \in \omega}\) and \((\beta_n)_{n \in \omega}\) of the elements of \( A \) such that:

(i) \( \alpha_0 = y \quad \beta_0 = x \),
(ii) \( \alpha_{n-1} \leq \alpha_n \) and \( \beta_{n-1} \geq \beta_n \) for \( n > 0 \),
(iii) \( \forall n \in \omega (\beta_{2n+1} \leq b_{2n+1} \lor \exists a \in B_{2n+1} b \leq \alpha_{n+1}) \),
\( \forall n \in \omega (\exists a \in A_{2n} \beta_{2n} \leq a \lor a_{2n} \leq \alpha_{2n}) \).
(iv) for every \( n \in \omega \) the relation \( \beta_n \leq \alpha_n \) does not hold.

Suppose that for \( k \in \omega \) \( \alpha_1, \ldots, \alpha_{2k} \) and \( \beta_1, \ldots, \beta_{2k} \) are constructed such that \( (ii) \) – \( (iv) \) are fulfilled. On account of \( (iv) \) we have that the relation \( \beta_{2k} \leq \alpha_{2k} \) does not hold. By Lemma 1 we have that for every \( c \in A \), the relations

\[
\beta_{2k} \leq \alpha_{2k} \cup c \text{ or } \beta_{2k} \cap c \leq \alpha_{2k}
\]
do not hold.

Putting \( c = b_{2k+1} \) we obtain that \( \beta_{2k} \leq \alpha_{2k} \cup b_{2k+1} \) does not hold or \( \beta_{2k} \cap b_{2k+1} \leq \alpha_{2k} \) does not hold.

Consider the first inequality. We have that

\[
\sim (\beta_{2k} \leq \alpha_{2k} \cup \bigcap_{b \in B_{2k+1}} b).
\]

By the condition \( (\bigcap, \bigcup) \) we infer that
\[ \sim (\beta_{2k} \leq \bigcap_{b \in B_{2k+1}} (\alpha_{2k} \cup b)), \]
i.e.

\[ \exists b \in B_{2k+1} \sim (\beta_{2k} \leq \alpha_{2k} \cup b). \]

Thus we have

\[ (\ast) \exists b \in B_{2k+1} \sim (\beta_{2k} \leq \alpha_{2k} \cup b) \text{ or } \sim (\beta_{2k} \cap b_{2k+1} \leq \alpha_{2k}). \]

Now if the first condition (\ast) is satisfied we put

\[ \alpha_{2k+1} = \alpha_{2k} \cup b \text{ and } \beta_{2k+1} = \beta_{2k}. \]

If the second condition (\ast) takes place we put

\[ \beta_{2k+1} = \beta_{2k} \cap b_{2k+1} \text{ and } \alpha_{2k+1} = \alpha_{2k}. \]

It is not difficult to check that so defined \( \alpha_{2k+1} \) and \( \beta_{2k+1} \) satisfy (ii) – (iv).

Having \( \alpha_{2k+1} \) and \( \beta_{2k+1} \) we define the \( \alpha_{2k+2} \) and \( \beta_{2k+2} \) in a similar way. By (iv) we have \( \beta_{2k+1} \leq \alpha_{2k+1} \) does not hold, i.e. that for every \( c \in A \), the relations

\[ \beta_{2k+1} \leq \alpha_{2k+1} \cup c \text{ or } \beta_{2k+1} \cap c \leq \alpha_{2k+1} \]
do not hold.

Putting \( c = a_{2k+2} \) we have that

\[ \sim (\beta_{2k+1} \leq \alpha_{2k+1} \cup a_{2k+2}) \text{ or } \exists a \in A_{2k+2} \sim (\beta_{2k+1} \cap a \leq \alpha_{2k+1}) \]

We define

\[ \beta_{2k+2} = \beta_{2k+1} \text{ and } \alpha_{2k+2} = \alpha_{2k+1} \cup a_{2k+2} \]
or

\[ \beta_{2k+2} = \beta_{2k+1} \cap a \text{ and } \alpha_{2k+2} = \alpha_{2k+1} \]

In both cases \( \alpha_{2k+2} \) and \( \beta_{2k+2} \) satisfied (ii) – (iv).

In this way we defined the sequences \( (\alpha_n)_{n \in \omega} \) and \( (\beta_n)_{n \in \omega} \). Let \( I \) be the ideal generated by the sequence \( (\alpha_n)_{n \in \omega} \) and \( F \) be the filter generated by the sequence \( (\beta_n)_{n \in \omega} \). By (iv) \( I \) and \( F \) are disjoint and

\[ (v) \forall n \in \omega (b_{2n+1} \in F \lor \exists b \in B_{2n+1} \ b \in I), \]

\[ (vi) \forall n \in \omega (a_{2n} \in I \lor \exists a \in A_{2n} \ a \in F). \]

It is well known that in a distributive lattice, every filter can be separated from an ideal, disjoint from it, by a prime filter. Let \( \nabla \) be a prime filter
containing $F$ such that $\nabla$ is disjoint from $I$. It is obvious that $x \in \nabla$ and $y \notin \nabla$. By (v) and (vi) $\nabla$ is the required $Q$-filter, which completes the proof of the theorem.

In the same way we can prove a condition necessary and sufficient for the existing of a prime ideal preserving enumerable infinite joins and meets in a distributive lattice.

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