A CHARACTERIZATION OF FRAGMENTS OF THE INTUITIONISTIC PROPOSITIONAL LOGIC

We shall use the symbols: $\rightarrow$, $\leftrightarrow$, $\land$, $\lor$, $\neg$ as the well-known connectives (implication, equivalence, conjunction, disjunction, negation). For every set of connectives $\Psi \subseteq \{\rightarrow, \leftrightarrow, \land, \lor, \neg\}$ by $F_\Psi$ we mean the set of formulas built up by means of propositional variables from an infinite set $V$ and the connectives from $\Psi$ (we shall write $F$ instead of operation $C$ in $F_\Psi$ is called $\Psi$-consequence (see [1]) iff the following conditions hold for every $X \subseteq F_\Psi$, $\alpha, \beta \in F_\Psi$:

1. ($\rightarrow$) if $\rightarrow \in \Psi$ then $C(X \cup \{\beta\}) \subseteq C(X \cup \{\alpha\})$ iff $\alpha \rightarrow \beta \in C(X)$,
2. ($\leftrightarrow$) if $\leftrightarrow \in \Psi$ then $C(X \cup \{\beta\}) = C(X \cup \{\alpha\})$ iff $\alpha \leftrightarrow \beta \in C(X)$,
3. ($\land$) if $\land \in \Psi$ then $C(\{\alpha, \beta\}) = C(\{\alpha \land \beta\})$,
4. ($\lor$) if $\lor \in \Psi$ then $C(X \cup \{\alpha\}) \cap C(X \cup \{\beta\}) = C(X \cup \{\alpha \lor \beta\})$,
5. ($\neg$) if $\neg \in \Psi$ then $C(X \cup \{\alpha\}) = F_\Psi$ iff $\neg \alpha \in C(X)$.

Let $Cn_\Psi$ denotes the consequence operation in $F$ determined by the theorems of the intuitionistic propositional logic and the detachment rule for the implication connective $\rightarrow$. Putting $Cn_\Psi(X) = F_\Psi \cap Cn(X)$ for every $X \subseteq F_\Psi$ one defines the consequence operation $Cn_\Psi$ in $F_\Psi$ (obviously $Cn = Cn_{\{\rightarrow, \leftrightarrow, \land, \lor, \neg\}}$). Grzegorczyk [1] proved that the intuitionistic consequence operation $Cn$ can be characterized as the smallest $\{\rightarrow, \leftrightarrow, \land, \lor, \neg\}$-consequence. We have the following generalization of Grzegorczyk’s results:

**Theorem.** For every $\Psi \subseteq \{\rightarrow, \leftrightarrow, \land, \lor, \neg\}$, the consequence operation $Cn_\Psi$ is the smallest $\Psi$-consequence.

The fact above can be viewed as a kind of separable characterization of fragments of the intuitionistic propositional logic (comp. [2]). We are
informed that a similar result was achieved independently by S. J. Surma (unpublished).

References


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