Bulletin of the **S**ection of **L**ogic
Volume 3/1 (1974), pp. 30–32

reedition 2012 [original edition, pp. 30–33]

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AN ALGEBRAIC CHARACTERIZATION OF THE NOTION OF STRUCTURAL COMPLETENESS

An extended version of this abstract will appear in Reports on Mathematical Logic.

Let $\mathcal{F} = \langle F, D\rangle$ be the free algebra in the class of all algebras of some fixed type free-generated by the set $F_c \subseteq F$. The algebra $\mathcal{F}$ will be called language, the operations from D – connectives, the elements of F – formulas and the elements of F_0 – variables. The subsets of $2^F \times F$ will be called rules of inference or shortly – rules. By propositional calculus we mean any pair $\Delta = \langle A, R\rangle$ where $A \subseteq F$ and R is a set of structural rules (see [2]). Every set of formulas containing A and closed with respect to each rule from R will be called Δ-system. The symbol C_Δ denotes the consequence operation in $\mathcal{F}$ determined by the calculus Δ (i.e. $C_\Delta(X)$ is the intersection of all Δ-systems containing X). The calculus Δ is called standard iff C_Δ is finite consequence operation (see [2]). It is known that for every standard calculus Δ, the union of arbitrary chain of Δ-systems is Δ-system and thus by Kuratowski-Zorn's lemma the condition $\alpha \notin C_\Delta(X)$ implies the existence of Δ-system Y such that $\alpha \notin Y \supseteq X$ and Y is a maximal Δ-system having that property. Every such Y will be called relatively maximal Δ-supersystem of X with respect to α. If Δ is standard calculus then by $RMS(\Delta)$ we denote the family of all relatively maximal Δ-supersystems (i.e. $Y \in RMS(\Delta)$ iff for some $X \subseteq F$ and $\alpha \in F$, Y is relatively maximal Δ-supersystem of X with respect to α). A rule is called Δ-permissible (Δ-derivable) iff $C_\Delta(\emptyset)$ (every Δ-system) is closed with respect to this rule. Following [1] we say that the calculus Δ is structurally complete iff every structural and Δ-permissible rule is Δ-derivable.

The matrix is a pair $\langle \mathcal{M}, Y \rangle$ where $\mathcal{M}$ is an algebra similar to $\mathcal{F}$ and X is a subset of the universe of $\mathcal{M}$. The common algebraic concepts like: homomorphism, isomorphism, embedding, congruence relation e.t.c are defined for matrices in [2]. We have the following general criterion of structural completeness for standard propositional calculi:

THEOREM 1. *If Δ is standard then it is structurally complete iff the following condition holds:* $(*)$ *for every $Y \in RMS(\Delta)$ there exists a homomorphism of the matrix $\langle \mathcal{F}, Y \rangle$ into $\langle \mathcal{F}, C_\Delta(\emptyset) \rangle$.*

Let $E \subseteq F$ be a set of formulas containing exactly two variables, say y and z, We will write $E(\alpha, \beta)$ to denote the set of all formulas which results by simultaneous substitution of α for y and β for z in some formula from E. The set E will be called Δ-equivalence iff for every $X \subseteq F$ the relation $\{\langle \alpha, \beta \rangle : E(\alpha, \beta) \subseteq C_\Delta(X)\}$ is a congruence of the matrix $\langle \mathcal{F}, C_\Delta(X) \rangle$. It is easy to see that E is Δ-equivalence iff the following conditions hold:

(i) $E(\alpha, \alpha) \subseteq C_\Delta(\emptyset)$,
(ii) $E(\alpha, \beta) \subseteq C_\Delta(E(\beta, \alpha))$,
(iii) $E)\alpha, \beta) \subseteq C_\Delta(E(\alpha, \gamma) \cup E(\gamma, \beta))$,
(iv) for every $d \in D$ if d is n-ary connective then $E(d(\alpha_1, \ldots, \alpha_n), d(\beta_1, \ldots, \beta_n)) \subseteq C_\Delta(E(\alpha_1, \beta_1) \cup \ldots \cup E(\alpha_n, \beta_n))$,
(v) $\alpha \in C_\Delta(E(\alpha, \beta)) \cup \{\beta\})$.

Note that if E and E' are Δ-equivalences then $C_\Delta(E(\alpha, \beta)) = C_\Delta(E'(\alpha, \beta))$ which gives that for every $X \subseteq F$ the congruence relations in $\langle \mathcal{F}, C_\Delta(X) \rangle$ obtained by means of E and E' are equal. Thus if Δ-equivalences exist then the congruence relation in $\langle \mathcal{F}, C_\Delta(X) \rangle$ obtainable by means of some Δ-equivalence is unique and will be denoted by $\equiv_X$. We call the calculus Δ equivalential iff some Δ-equivalence exists.

THEOREM 2. *If Δ is standard and equivalential then it is structurally complete iff the following condition holds:* $(**)$ *for every $Y \in RMS(\Delta)$ there exists an embedding of the matrix $\langle \mathcal{F}/\equiv_Y, Y/\equiv_Y \rangle$ into $\langle \mathcal{F}/\equiv_\emptyset, C_\Delta(\emptyset)/\equiv_\emptyset \rangle$.*

Observe that if Δ is equivalential, Y and X are Δ-systems and h is a homomorphism of the matrix $\langle \mathcal{F}, Y \rangle$ into $\langle \mathcal{F}, X \rangle$ then the mapping $g : [\alpha]\equiv_Y \to [h(\alpha)]\equiv_X$ is an embedding of $\langle \mathcal{F}/\equiv_Y, Y/\equiv_Y \rangle$ into $\langle \mathcal{F}/\equiv_X, X/\equiv_X \rangle$ and therefore Theorem 2 can be obtained from Theorem 1 as a corollary.

References

[1] W. A. Pogorzelski, *Structural Completeness of the Propositional Calculus*, **Bulletin de l'Academie Polonaise des Sciences**, Série des sciences mathematiques, astronomiques et physiques, 19, No 5 (1971), pp. 349–351.

[2] R. Wójcicki, *Matrix Approach in Methodology of Sentential Calculi*, **Studia Logica** 52 (1973), pp. 7–37.

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