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BINARY FUNCTIONS DEFINABLE IN IMPLICATIONAL GOEDEL ALGEBRAS

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The notion of GOEDEL ALGEBRA was introduced in [1]. In what follows we shall deal with the PURE IMPLICATIONAL Goedel algebras only. By G^i_n we shall mean the algebra $\langle\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}, \to\rangle$, where the operation $\to$ is defined as follows:

$$x \to y = \begin{cases} 1 & \text{if } x \leqslant y \\ y & \text{otherwise.} \end{cases}$$

By the n-valued GOEDEL MATRIX we shall mean the matrix $\underline{G}^i_n = \langle G^i_n, \{1\}\rangle$. Let us denote by $\underline{L}$ the suitable sentential language for those matrices, i.e. $\underline{L} = \langle L, \to\rangle$, where $\to$ is a binary connective and L is the set of formulas built up by means of the sentential variables $p_1, p_2, \ldots$ and the connective $\to$. The valuations of the formulas of $\underline{L}$ in $\underline{G}^i_n$ will be denoted by $h; h : \underline{L} \to^{hom} G^i_n$. Where $\alpha(p, q, \ldots, r)$ is a formula of $\underline{L}$ built up by means of exactly the variables $p, q, \ldots, r$, we shall denote by $\alpha(x, y, \ldots, z)$ the value of α under the valuation h such that $hp = x$, $hq = y, \ldots, hr = z$, i.e. $\alpha(x, y, \ldots, z) = h(\alpha(p, q, \ldots, r))$.

Definition 1. A function $f(x_1, \ldots, x_k)$, $f : \{0, \frac{1}{n-1}, \ldots, 1\}^k \to \{0, \frac{1}{n-1}, \ldots, 1\}$, is DEFINABLE in the algebra G^i_n if there is such a formula $\alpha(p_1, \ldots, p_k)$ of $\underline{L}$ that for every sequence $x^0_1, \ldots, x^0_k$ of elements of G^i_n the equality holds

$$f(x^0_1, \ldots, x^0_k) = \alpha(x^0_1, \ldots, x^0_k).$$

We shall say in such a case that α DEFINES the function f.

All the theorems of this abstract will be stated without any proof.

LEMMA. *If $f = f(x_1, \ldots, x_k)$ is definable implicational Goedel algebra then there exists such an index i $(1 \leqslant i \leqslant k)$ that for every sequence $x_1^0, \ldots, x_k^0$ of elements of that algebra*

$$f(x_1^0, \ldots, x_k^0) \in \{x_i^0, 1\}.$$

DEFINITION 2. We shall say that the function $f(x_1, \ldots, x_k)$ DEPENDS MAINLY on the variable x_i if i fulfils the conclusion of the Lemma.

THEOREM 1. *Let $f(x, y), f : \{0, \frac{1}{n-1}, \ldots, 1\}^2 \to \{0, \frac{1}{n-1}, \ldots, 1\}$, be a binary function which depends mainly on y. Then f is definable in G_n^i if and only if the following conditions are satisfied:*

1. *Let a, b be arbitrary elements of G_n^i not equal to 1. Then $f(a, a) = a$ iff $f(b, b) = b$.*
2. *Let a, b be arbitrary elements of G_n^i not equal to 1. Then $f(1, a) = a$ iff $f(1, b) = b$.*
3. *Let a, b, $a > b$, be arbitrary elements of G_n^i not equal to 1. Then $f(a, b) = 1$ iff $f(1, b) = 1$.*
4. *Let a, b, c, d, $a < b$, $c < d$, be arbitrary elements of G_n^i not equal to 1. Then $f(a, b) = b$ iff $f(c, d) = d$.*

COROLLARY 1. *For any $n > 2$ there exist exactly 14 binary functions definable in G_n^i.*

It is possible to give a short proof of a special case of general theorem of Wroński [2], on the grounds of Theorem 1:

COROLLARY 2. *The set $E(\underline{G}_3^i)$ is not axiomatizable by means of formulas built up of two variables only.*

References

[1] K. Goedel, *Zum intutionistischen Aussagenkalkül*, Akademie der Wissenschaften in Wien, Mathematisch-naturwissenschaftliche Klasse, Anzeiger, 69 (1932), pp. 65–66.

[2] A. Wroński, *Axiomatization of the implicational Goedel's matrices by Kalmar's method*, [in:] **Studies in the history of mathematical logic**, Ossolineum, Wrocław-Warszawa-Kraków-Gdańsk 1973, pp. 123–132.

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