PARTIAL BOOLEAN \(\sigma\)-ALGEBRAS

The present note contains some simple generalizations of notions introduced in [1] and [2]. We refer the reader to those papers for all relevant definitions.

**Definition 1.** We shall say that the system \(\mathcal{B} = (B; \bot; \lor; \neg; 1)\) is a partial Boolean \(\sigma\)-algebra (\(PB\sigma\) Algebra) if \(\mathcal{B}\) is a partial Boolean algebra and for every denumerable sequence \(\{a_n\}_{n \in \mathbb{N}}\) of mutually commeasurable elements of \(B(a_m \bot a_n)\) for \(m, n \in \mathbb{N}\) there exists \(b \in B\) that

\[
\begin{align*}
(i) & \quad \forall_{n \in \mathbb{N}} a_n \subseteq b \text{ (i.e., } a_n \bot b \text{ and } a_n \lor b = b) \\
(ii) & \quad \forall_{c \in B} \forall_{n \in \mathbb{N}} (a_n \bot c \Rightarrow b \bot c) \\
(iii) & \quad \forall_{c \in B} \forall_{n \in \mathbb{N}} (a_n \subseteq c \Rightarrow b \subseteq c)
\end{align*}
\]

We shall write: \(b = \bigvee_{n=1}^{\infty} a_n\). \(\bigvee\) is a well-defined partial, infinite function.

**Example 1.** Let \(L(H)\) be the set of all closed subspaces of a separable Hilbert space. It is well-known that \(L(H) = (L(H); \bot; \lor; \neg; H)\) is a \(PA\) Algebra, where \(\bot\) is defined as follows:

\(H_1 \bot H_2\) iff there exist a basis \(\varepsilon\) and subsets \(\varepsilon_i \subseteq \varepsilon(i = 1, 2)\) such that \(\varepsilon_i\) is a basis in \(H_i(i = 1, 2)\) and \(\neg H_0\) is the orthogonal complementation of \(H_0([1])\).

\(L(H)\) is also a \(PB\sigma\) Algebra because of the following simple fact:

If \(H_1, H_2, \ldots, H_n, \ldots\) are mutually commeasurable \((H_m \bot H_n\) for \(m, n \in \mathbb{N}\)) then there exists a basis \(\varepsilon\) in \(H\) and subsets \(\varepsilon_n \subseteq \varepsilon(n = 1, 2, \ldots)\) such that \(\varepsilon_n\) is a basis in \(H_n(n = 1, 2, \ldots)\).
As before one can consider the logic based on $PB\sigma$Algebras. Strictly speaking, let $P_0 = \{ x_\nu : \nu < \omega_1 \}$ be the set of variables. $P$ – the set of formulas of infinite length – is defined as the least set satisfying the conditions:

(1) $P_0 \subseteq P$
(2) If $\varphi_1, \varphi_2, \ldots \in P$, then $\bigvee \{ \varphi_n : n \in N \} \in P$
(3) If $\varphi \in P$, then $\neg \varphi \in P$.

By a valuation $h$ of $P$ in $B$ ($B \in PB\sigma A$) we shall mean every partial homomorphism of $P_0$ into $B$, i.e., a function whose domain $\text{Dom}(h)$ is the least set, such that:

(a) Some variables belong to $\text{Dom}(h)$ (but not necessarily all); if $x_\nu \in \text{Dom}(h)$, then $hx_\nu \in B$.
(b) If $\varphi_1, \varphi_2, \ldots \in \text{Dom}(h)$ and $h\varphi_1, h\varphi_2, \ldots$ are mutually comeasurable, then $\bigvee \{ \varphi_n : n \in N \} \in \text{Dom}(h)$; $h(\bigvee \{ \varphi_n : n \in N \}) = \bigvee_{n=1}^{\infty} h\varphi_n$.
(c) If $\varphi \in \text{Dom}(h)$, then $\neg \varphi \in \text{Dom}(h)$; $h\neg \varphi = \neg (h\varphi)$.

By a model $M$ we shall mean any pair $M = \langle B, h \rangle$, where $B \in PB\sigma A$ and $h$ is a valuation $h : P \to B$. We shall say that $\varphi$ is valid in $M = \langle B, h \rangle$ if $\varphi \in \text{Dom}(h)$ and $h\varphi = 1$. A formula $\varphi$ holds in $B$ ($B \in PB\sigma A$) if $h\varphi = 1$ for every valuation $h : P \to B$, such that $\varphi \in \text{Dom}(h)$. Similarly, we shall say that an identity $\varphi = \phi$ holds in $B$ ($\varphi, \phi \in P$) if $h\varphi = j\phi$ for every valuation $h (h : P \to B)$, such that $\varphi, \phi \in \text{Dom}(h)$. A formula $\varphi$ is valid if it holds in every $PB\sigma$Algebra.

It is easy now to define a consequence operation $Cn$ on $P$. Let $\alpha \in P$ and $X \subseteq P$. Then $\alpha \in Cn(X)$ if for every $PB\sigma$Algebra $B$ and every valuation $h : P \to B$ it holds:

$$\forall \beta \in X (\beta \in \text{Dom}(h) \Rightarrow h\beta = 1) \Rightarrow (\alpha \in \text{Dom}(h) \Rightarrow h\alpha = 1).$$

Let us observe that $Cn(\emptyset)$ is the set of all valid formulas. In a similar way one can construct the logic based on $PBA$Algebras. Then $Cn$ is a structural consequence operation.

As in the case of $PBA$Algebras it is easy to axiomatize $Cn(\emptyset)$ by a simple modification of methods of Kochen and Specker [1].

It is interesting to establish connections between $PB\sigma$Algebras and quantum logic (i.e., weakly modular orthoposets). Let us recall this notion:
\( L = \langle L; \perp; \subseteq; 1 \rangle \) is a quantum Logic if it is a finitely additive quantum logic (see [2], p. 168) and satisfies the condition:

\((L1)^*\) For every denumerable sequence \( \{a_n\}_{n \in \mathbb{N}} \) of mutually orthogonal elements of \( L \) there exists the least upper bound \( \bigvee_{n=1}^{\infty} a_n \).

A partial Boolean \( \sigma \)-algebra is transitive if it is transitive as PB\( \sigma \)Algebra (see [2], p. 167).

**Theorem 1.** (a) Let \( B = \langle B; \perp; \vee; \neg; 1 \rangle \) be a transitive partial Boolean \( \sigma \)-algebra. Then \( B^* = \langle B; \subseteq; \perp; 1 \rangle \), where \( a^\perp = \neg a \), is a quantum Logic.

(b) Let \( L = \langle L; \subseteq; \perp; 1 \rangle \) be a quantum logic. Then \( L^0 = \langle ; \leftrightarrow; \vee; \neg; 1 \rangle \) is a transitive PB\( \sigma \)Algebra, where \( \neg a = a^\perp \) and \( a \vee b \) is the l.u.b. of \( a \) and \( b \) \( (a \leftrightarrow b) \).

(c) Transitive partial Boolean \( \sigma \)-algebras \( B \) and \( B^* \) are identical.

(d) Quantum logics \( L \) and \( L^0 \) are identical.

Partial Boolean \( \sigma \)-algebras are generalizations of quantum logics (because there exist PB\( \sigma \)Algebras which are not inbeddable into any transitive PB\( \sigma \)Algebra). It would be interesting from physical point of view to develope a calculus of observables just in partial Boolean \( \sigma \)-algebras. Connections with some physical problems like Hidden-Variable Theories are easily seen in the following Imbedding Theorems:

**Theorem 2a.** Let \( L \in PB\sigma A \). The following conditions are equivalent:

(i) There exists a homomorphism of \( L \) into a Boolean \( \sigma \)-algebra.

(ii) No formula valid in all Boolean \( \sigma \)-algebras is refutable in \( L \).

(iii) For every denumerable set \( A = \{a_1, a_2, \ldots \} \subseteq L \) there exists a sequence \( (\varepsilon_1, \varepsilon_2, \ldots), \varepsilon_n = \begin{cases} 0 \\ 1 \end{cases} \), such that for every set of mutually commensurable elements \( \{a_{k_1}, a_{k_2}, \ldots \} \subseteq A \) the inequality holds:

\[
\bigwedge_{m=1}^{\infty} a_{\varepsilon_m}^{k_m} \neq 0.
\]

**Theorem 2b.** Let \( L \in PB\sigma A \). The following conditions are equivalent:

(i) \( L \) can be weakly imbedded into a Boolean \( \sigma \)-algebra.

(ii) Each formula valid in all Boolean \( \sigma \)-algebras holds in \( L \).
(iii) For every denumerable set $A = \{a_1, a_2, \ldots \} \subseteq L$ and every element $a_{i_0} \in A \ (a_{i_0} \neq 0)$ there exists a sequence $(\varepsilon_1, \varepsilon_2, \ldots)$ such that
1. $\varepsilon_{i_0} = 1$.
2. For every set of mutually commeasurable elements $\{a_{k_1}, a_{k_2}, \ldots \} \subseteq A$ the inequality holds: $\bigwedge_{m=1}^{\infty} a_{k_m}^{\varepsilon_{km}} \neq 0$.

Theorem 2c. Let $L \in \mathbb{P}B\sigma A$. The following conditions are equivalent:

(i) $L$ can be imbedded into a Boolean $\sigma$-algebra.

(ii) Each identity $\varphi = \phi$, which holds in all Boolean $\sigma$-algebras, is valid in $L$.

(iii) For every denumerable set $A = \{a_1, a_2, \ldots \} \subseteq L$ and every two distinct elements $a_{i_1}, a_{i_2} \in A$ there exists a sequence $(\varepsilon_1, \varepsilon_2, \ldots)$ such that:
1. either $\varepsilon_{i_1} = 1, \varepsilon_{i_2} = 0$ or $\varepsilon_{i_1} = 0, \varepsilon_{i_2} = 1$
2. for every set of mutually commeasurable elements $\{a_{k_1}, a_{k_2}, \ldots \} \subseteq A$ the inequality holds: $\bigwedge_{m=1}^{\infty} a_{k_m}^{\varepsilon_{km}} \neq 0$.

References
