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Janusz Czelakowski

PARTIAL BOOLEAN σ-ALGEBRAS

The present note contains some simple generalizations of notions introduced in [1] and [2]. We refer the reader to those papers for all relevant definitions.

Definition 1. We shall say that the system $\underline{B} = \langle B; \overset{|}{\circ}; \vee, \neg; 1\rangle$ is a partial Boolean σ-algebra ($PB\sigma$Algebra) if $\underline{B}$ is a partial Boolean algebra and for every denumerable sequence $\{a_n\}_{n\in N}$ of mutually commeasurable elements of $B(a_m \overset{|}{\circ} a_n$ for $m, n \in N)$ there exists $b \in B$ that

(i) $\forall_{n\in N} a_n \subseteq b$ (i.e., $a_n \overset{|}{\circ} b$ and $a_n \vee b = b$)

(ii) $\forall_{c\in B}\forall_{n\in N}(a_n \overset{|}{\circ} c \Rightarrow b \overset{|}{\circ} c)$

(iii) $\forall_{c\in B}\forall_{n\in N}(a_n \subseteq c \Rightarrow b \subseteq c)$

We shall write: $b = \bigvee_{n=1}^{\infty} a_n$. $\bigvee$ is a well-defined partial, infinite function.

Example 1. Let $L(H)$ be the set of all closed subspaces of a separable Hilbert space. It is well-known that $\underline{L}(H) = \langle L(H); \overset{|}{\circ}; \vee; \neg; H\rangle$ is a PAAlgebra, where $\overset{|}{\circ}$ is defined as follows:

$H_1 \overset{|}{\circ} H_2$ iff there exist a basis ε and subsets
$\varepsilon_i \subseteq \varepsilon (i = 1, 2)$ such that ε_i is a basis in $H_i (i = 1, 2)$

and $\neg H_0$ is the orthogonal complementation of H_0 ([1]).
$\underline{L}(H)$ is also a $PB\sigma$Algebra because of the following simple fact:

If $H_1, H_2, \ldots, H_n, \ldots$ are mutually commeasurable ($H_m \overset{|}{\circ} H_n$ for $m, n \in N$) then there exists a basis ε in H and subsets $\varepsilon_n \subseteq \varepsilon (n = 1, 2, \ldots)$ such that ε_n is a basis in $H_n (n = 1, 2, \ldots)$

As before one can consider the logic based on $PB\sigma$Algebras. Strictly speaking, let $P_0 = \{x_\nu : \nu < \omega_1\}$ be the set of variables. P – the set of formulas of infinite length-is defined as the least set satisfying the conditions:

(1) $P_0 \subseteq P$
(2) If $\varphi_1, \varphi_2, \ldots \in P$, then $\bigvee\{\varphi_n : n \in N\} \in P$
(3) If $\varphi \in P$, then $\neg\varphi \in P$.

By a valuation h of $\underline{P}$ in $\underline{B}$ ($\underline{B} \in PB\sigma A$) we shall mean every partial homomorphism of $\underline{P}$ into $\underline{B}$, i.e., a function whose domain $Dom(h) \subseteq P$ ($h : Dom(h) \to B$) is the least set, such that:

(a) Some variables belong to $Dom(h)$ (but not necessarily all); if $x_\nu \in Dom(h)$, then $hx_\nu \in B$.
(b) If $\varphi_1, \varphi_2, \ldots \in Dom(h)$ and $h\varphi_1, h\varphi_2, \ldots$ are mutually comeasurable, then $\bigvee\{\varphi_n : n \in N\} \in Dom(h); h(\bigvee\{\varphi_n : n \in N\}) = \bigvee_{n=1}^{\infty} h\varphi_n$.
(c) If $\varphi \in Dom(h)$, then $\neg\varphi \in Dom(h)$; $h\neg\varphi = \neg(h\varphi)$.

By a model $\mathcal{M}$ we shall mean any pair $\mathcal{M} = \langle \underline{B}, h\rangle$, where $\underline{B} \in PB\sigma A$ and h is a valuation ($h : \underline{P} \to \underline{B}$). We shall say that φ is valid in $\mathcal{M} = \langle \underline{B}, h\rangle$ if $\varphi \in Dom(h)$ and $h\varphi = 1$. A formula φ holds in $\underline{B}$ ($\underline{B} \in PB\sigma A$) if $h\varphi = 1$ for every valuation $h : \underline{P} \to \underline{B}$, such that $\varphi \in Dom(h)$. Similarly, we shall say that an identity $\varphi = \phi$ holds in $\underline{B}$ ($\varphi, \phi \in P$) if $h\varphi = j\phi$ for every valuation h ($h : \underline{P} \to \underline{B}$), such that $\varphi, \phi \in Dom(h)$. A formula φ is valid if it holds in every $PB\sigma$Algebra.

It is easy now to define a consequence operation Cn on $\underline{P}$. Let $\alpha \in P$ and $X \subseteq P$. Then $\alpha \in Cn(X)$ iff for every $PB\sigma$Algebra $\underline{B}$ and every valuation $h : \underline{P} \to \underline{B}$ it holds:

$$\forall_{\beta \in X}(\beta \in Dom(h) \Rightarrow h\beta = 1) \Rightarrow (\alpha \in Dom(h) \Rightarrow h\alpha = 1).$$

Let us observe that $Cn(\emptyset)$ is the set of all valid formulas. In a similar way one can construct the logic based on PBAlgebras. Then Cn is a structural consequence operation.

As in the case of PBAlgebras it is easy to axiomatize $Cn(\emptyset)$ by a simple modification of methods of Kochen and Specker [1].

It is interesting to establish connections between $PB\sigma$Algebras and quantum logic (i.e., weakly modular orthoposets). Let us recall this notion:

$\underline{L} = \langle L;^{\perp};\subseteq;1\rangle$ is a quantum Logic if it is a finitely additive quantum logic (see [2], p. 168) and satisfies the condition:

$(L1)^*$ For every denumerable sequence $\{a_n\}_{n\in N}$ of mutually orthogonal elements of L there exists the least upper bound $\bigvee_{n=1}^{\infty} a_n$.

A partial Boolean σ-algebra is transitive if it is transitive as PBAlgebra (see [2], p.167).

THEOREM 1. *(a) Let* $\underline{B} = \langle B; \overset{|}{\circ}; \vee; \neg; 1\rangle$ *be a transitive partial Boolean* σ*-algebra. Then* $\underline{B}^* = \langle B;\subseteq;\perp;1\rangle$*, where* $a^{\perp} = \neg a$*, is a quantum Logic.*

(b) Let $\underline{L} = \langle L;\subseteq;\perp;1\rangle$ *be a quantum logic. Then* $L^0 = \langle ;\leftrightarrow;\vee,\neg;1\rangle$ *is a transitive* $PB\sigma$*Algebra, where* $\neg a = a^{\perp}$ *and* $a \vee b$ *is the l.u.b. of* a *and* b $(a \leftrightarrow b)$.

(c) Transitive partial Boolean σ*-algebras* $\underline{B}$ *and* $\underline{B}^{*0}$ *are identical.*

(d) Quantum logics $\underline{L}$ *and* $\underline{L}^{0*}$ *are identical.*

Partial Boolean σ-algebras are generalizations of quantum logics (because there exist $PB\sigma$Algebras which are not inbeddable into any transitive $PB\sigma$Algebra). It would be interesting from physical point of view to develope a calculus of observables just in partial Boolean σ-algebras. Connections with some physical problems like Hidden-Variable Theories are easily seen in the following Imbedding Theorems:

THEOREM 2A. *Let* $\underline{L} \in PB\sigma A$*. The following conditions are equivalent:*

(i) There exists a homomorphism of $\underline{L}$ *into a Boolean* σ*-algebra.*

(ii) No formula valid in all Boolean σ*-algebras is refutable in* $\underline{L}$*.*

(iii) For every denumerable set $A = \{a_1, a_2, \ldots\} \subseteq L$ *there exists a sequence* $(\varepsilon_1, \varepsilon_2, \ldots)$, $\varepsilon_n = \begin{cases} 0 \\ 1 \end{cases}$, *such that for every set of mutually commeasurable elements* $\{a_{k_1}, a_{k_2}, \ldots\} \subseteq A$ *the inequality holds;* $\bigwedge_{m=1}^{\infty} a_{k_m}^{\varepsilon_{k_m}} \neq 0$.

THEOREM 2B. *Let* $\underline{L} \in PB\sigma A$*. The following conditions are equivalent:*

(i) $\underline{L}$ *can be weakly imdedded into a Boolean* σ*-algebra.*

(ii) Each formula valid in all Boolean σ*-algebras holds in* $\underline{L}$*.*

(iii) For every denumerable set $A = \{a_1, a_2, \ldots\} \subseteq L$ *and every element* $a_{i_0} \in A$ $(a_{i_0} \neq 0)$ *there exists a sequence* $(\varepsilon_1, \varepsilon_2, \ldots)$ *such that*

1. $\varepsilon_{i_0} = 1$.
2. *For every set of mutually commeasurable elements* $\{a_{k_1}, a_{k_2}, \ldots\} \subseteq A$ *the inequality holds:* $\bigwedge_{m=1}^{\infty} a_{k_m}^{\varepsilon_{k_m}} \neq 0$.

THEOREM 2C. *Let* $\underline{L} \in PB\sigma A$. *The following conditions are equivalent:*

(i) $\underline{L}$ *can be imbedded into a Boolean* σ*-algebra.*

(ii) Each identity $\varphi = \phi$, *which holds in all Boolean* σ*-algebras, is valid in* $\underline{L}$.

(iii) For every denumerable set $A = \{a_1, a_2, \ldots\} \subseteq L$ *and every two distinct elements* $a_{i_1}, a_{i_2} \in A$ *there exists a sequence* $(\varepsilon_1, \varepsilon_2, \ldots)$ *such that:*

1. *either* $\varepsilon_{i_1} = 1$, $\varepsilon_{i_2} = 0$ *or* $\varepsilon_{i_1} = 0$, $\varepsilon_{i_2} = 1$
2. *for every set of mutually commeasurable elements* $\{a_{k_1}, a_{k_2}, \ldots\} \subseteq A$ *the inequality holds:* $\bigwedge_{m=1}^{\infty} a_{k_m}^{\varepsilon_{k_m}} \neq 0$.

References

[1] S. Kochen and E. P. Specker, *Logical structure aristing in quantum theory*, [in:] **The theory of models**, ed. by J. W. Addison, L. Henkin and A. Tarski, North-Holland, Amsterdam, 1965.

[2] J. Czelakowski, *Some remarks on transitive partial Boolean algebras*, **Bulletin of the Section of Logic of Inst. of Phil. and Soc. Pol. Acad. Sci.**, vol. 2, No. 3.