Bulletin of the **S**ection of **L**ogic
Volume 3/1 (1974), pp. 43–44

reedition 2012 [original edition, pp. 43–44]

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FINITE LIMITATIONS ON A PROPOSITIONAL CALCULUS FOR AFFIRMATION AND NEGATION

Let A be a distributive lattice with $0, 1$ and let B_0 be the Boolean algebra of complemented elements of A. A is called pseudo-supplemented if for each $x \in A$ there is a largest element $!x$ in B_0 such that $!x \leqslant x$. A is called a B algebra if for each x, y in A there is a largest element $x \Rightarrow y$ in B_0 such that $x(x \Rightarrow y) \leqslant y$. B algebras which satisfy the identity $(x \Rightarrow y) + (y \Rightarrow x) = 1$ are called BL algebras. BL algebras are the algebraic systems corresponding to the propositional calculi for affirmation and negation discussed in our previous abstract (JSL, vol. 37, No. 2, June 1972, p. 439).

Post algebras of order n are important special cases of BL algebras. The following conditions are equivalent for a B algebra A. (1) A is a B subalgebra of a Post algebra of order n; (2) A satisfies the identity $(1 \Rightarrow x_1) + (x_1 \Rightarrow x_2) + \ldots + (x_{n-2} \Rightarrow x_{n-1}) + (x_{n-1} \Rightarrow 0) = 1$; (3) A is a BL algebra and every chain of prime filters in A has at most $n-1$ elements; (4) A is a pseudo-supplemented Heyting algebra satisfying $!(x+y) = !x + !y$ and $(1 \to x_1) + (x_1 \to x_2) + \ldots + (x_{n-2} \to x_{n-1}) + (x_{n-1} \to 0) = 1$, where $\to$ is the arrow connective of a Heyting algebra.

The case $n = 3$ is of interest from the standpoint of computer science and semantics. Let S be a pseudo-supplemented Stone lattice. An element $x \in S$ is called dense if $\neg x = 0$; an element $y \in S$ is called sparse if $!y = 0$. The following are equivalent for such a lattice S: (5) S is a B subalgebra of a Post algebra of order 3; (6) every sparse element of S is less than or equal to every dense element of S; (7) for any $x \in S$ and $y \in S$, $x\neg!x\neg\neg y \leqslant y$. If S is finite, then under any one of these conditions (5), (6), or (7), there will be an element e such that (*) $x = e\neg\neg x + !x$ for all $x \in S$. Whether S is finite or not, if there exists an element e which is dense such that (8) holds, then S is a direct product of Post algebras which have a maximum order of 3 and so is a B algebra A satisfying condition (1), (2), (3), or (4); if there exists an element e which is both dense and sparse such that (8) holds, then S is a Post algebra of order 3.